Chapter 3
Basic Plasma Physics

3.1 Introduction

Electric propulsion achieves high specific impulse by the acceleration of charged particles to high velocity. The charged particles are produced by ionization of a propellant gas, which creates both ions and electrons and forms what is called a plasma. Plasma is then a collection of the various charged particles that are free to move in response to fields they generate or fields that are applied to the collection and, on the average, is almost electrically neutral. This means that the ion and electron densities are nearly equal, $n_i \approx n_e$, a condition commonly termed "quasi-neutrality." This condition exists throughout the volume of the ionized gas except close to the boundaries, and the assumption of quasi-neutrality is valid whenever the spatial scale length of the plasma is much larger than the characteristic length over which charges or boundaries are electrostatically shielded, called the Debye length. The ions and electrons have distributions in energy usually characterized by a temperature T_i for ions and T_e for electrons, which are not necessarily or usually the same. In addition, different ion and electron species can exist in the plasma with different temperatures or different distributions in energy.

Plasmas in electric propulsion devices, even in individual parts of a thruster, can span orders of magnitude in plasma density, temperature, and ionization fraction. Therefore, models used to describe the plasma behavior and characteristics in the thrusters must be formed with assumptions that are valid in the regime being studied. Many of the plasma conditions and responses in thrusters can be modeled by fluid equations, and kinetic effects are only important in specific instances.

There are several textbooks that provide very comprehensive introductions to plasma physics [1–3] and the generation of ion beams [4]. This chapter is intended to provide the basic plasma physics necessary to understand the operation of ion and Hall thrusters. The units used throughout the book are based on the International System (SI). However, by convention we will occasionally revert to other metric units (such as A/cm^2, mg/s, etc.) commonly used in the literature describing these devices.

3.2 Maxwell's Equations

The electric and magnetic fields that exist in electric propulsion plasmas obey Maxwell's equations formulated in a vacuum that contains charges and currents. Maxwell's equations for these conditions are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_o} \tag{3.2-1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{3.2-2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.2-3}$$

$$\nabla \times \mathbf{B} = \mu_o \left(\mathbf{J} + \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} \right), \tag{3.2-4}$$

where ρ is the charge density in the plasma, $\mathbf{J}$ is the current density in the plasma, and ε_o and μ_o are the permittivity and permeability of free space, respectively. Note that ρ and $\mathbf{J}$ comprise all the charges and currents for all the particle species that are present in the plasma, including multiply charged ions. The charge density is then

$$\rho = \sum_s q_s n_s = e\left(Z n_i - n_e\right), \tag{3.2-5}$$

where q_s is the charge state of species s, Z is the charge state, n_i is the ion number density, and n_e is the electron number density. Likewise, the current density is

$$J = \sum_s q_s n_s v_s = e\left(Z n_i v_i - n_e v_e\right), \tag{3.2-6}$$

where v_s is the velocity of the charge species, v_i is the ion velocity, and v_e is the electron velocity. For static magnetic fields $\left(\partial \mathbf{B}/\partial t = 0\right)$, the electric field can be expressed as the gradient of the electric potential,

$$\mathbf{E} = -\nabla\phi\,, \tag{3.2-7}$$

where the negative sign comes from the convention that the electric field always points in the direction of ion motion.

3.3 Single Particle Motions

The equation of motion for a charged particle with a velocity **v** in a magnetic field **B** is given by the Lorentz force equation:

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right). \tag{3.3-1}$$

Particle motion in a magnetic field in the $\hat{z}$ direction for the case of negligible electric field is found by evaluating Eq. (3.3-1):

$$\begin{aligned} m\frac{\partial v_x}{\partial t} &= qBv_y \\ m\frac{\partial v_y}{\partial t} &= -qBv_x \\ m\frac{\partial v_z}{\partial t} &= 0. \end{aligned} \tag{3.3-2}$$

Taking the time derivative of Eq. (3.3-2) and solving for the velocity in each direction gives

$$\begin{aligned} \frac{\partial^2 v_x}{\partial t^2} &= \frac{qB}{m}\frac{\partial v_y}{\partial t} = -\left(\frac{qB}{m}\right)^2 v_x \\ \frac{\partial^2 v_y}{\partial t^2} &= -\frac{qB}{m}\frac{\partial v_x}{\partial t} = -\left(\frac{qB}{m}\right)^2 v_y. \end{aligned} \tag{3.3-3}$$

These equations describe a simple harmonic oscillator at the cyclotron frequency:

$$\omega_c = \frac{|q|\,B}{m}\,. \tag{3.3-4}$$

For electrons, this is called the electron cyclotron frequency.

The size of the particle orbit for finite particle energies can be found from the solution to the particle motion equations in the axial magnetic field. In this case, the solution to Eq. (3.3-3) is

$$v_{x,y} = v_{\perp} e^{i\omega_c t} \, . \tag{3.3-5}$$

The equation of motion in the y-direction in Eq. (3.3-2) can be rewritten as

$$v_y = \frac{m}{qB} \frac{\partial v_x}{\partial t} = \frac{1}{\omega_c} \frac{\partial v_x}{\partial t} \, . \tag{3.3-6}$$

Utilizing Eq. (3.3-5), Eq. (3.3-6) becomes

$$v_y = \frac{1}{\omega_c} \frac{\partial v_x}{\partial t} = i v_{\perp} e^{i\omega_c t} = \frac{\partial y}{\partial t} \, . \tag{3.3-7}$$

Integrating this equation gives

$$y - y_o = \frac{v_{\perp}}{\omega_c} e^{i\omega_c t} \, . \tag{3.3-8}$$

Taking the real part of Eq. (3.3-8) gives

$$y - y_o = \frac{v_{\perp}}{\omega_c} \cos \, \omega_c t = r_L \, \cos \, \omega_c t \, , \tag{3.3-9}$$

where $r_L = v_{\perp} / \omega_c$ is defined as the Larmor radius. A similar analysis of the displacement in the $\hat{x}$ direction gives the same Larmor radius 90 degrees out of phase with the $\hat{y}$-direction displacement, which then with Eq. (3.3-9) describes the particle motion as a circular orbit around the field line at x_o and y_o with a radius given by r_L .

The Larmor radius arises from very simple physics. Consider a charged particle of mass, *m*, in a uniform magnetic field with a velocity in one direction, as illustrated in Fig. 3-1. The charge will feel a Lorentz force

$$\mathbf{F} = q\mathbf{v}_{\perp} \times \mathbf{B} \, . \tag{3.3-10}$$

Since the charged particle will move under this force in circular orbits in the $\mathbf{v}_{\perp} \times \mathbf{B}$ direction, it feels a corresponding centripetal force such that

$$\mathbf{F_c} = q\mathbf{v}_\perp \times \mathbf{B} = \frac{mv_\perp^2}{r}, \tag{3.3-11}$$

where r is the radius of the cycloidal motion in the magnetic field. Solving for the radius of the circle gives

$$r = r_L = \frac{mv_\perp}{qB}, \tag{3.3-12}$$

which is the Larmor radius.

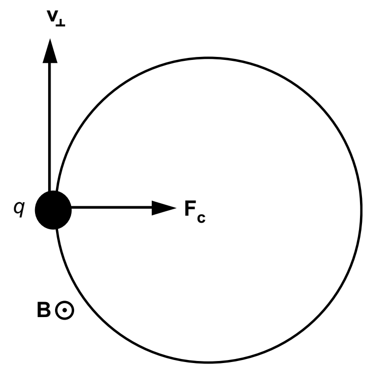

Fig. 3-1. Positively charged particle moving in a uniform vertical magnetic field.

The Larmor radius can be written in a form simple to remember:

$$r_L = \frac{v_\perp}{\omega_c} = \frac{1}{B}\sqrt{\frac{2mV_\perp}{e}}, \tag{3.3-13}$$

using $1/2\, mv_\perp^2 = eV_\perp$ for the singly charged particle energy in the direction perpendicular to the magnetic field. The direction of particle gyration is always such that the induced magnetic field is opposite in direction to the applied field, which tends to reduce the applied field, an effect called diamagnetism. Any particle motion along the magnetic field is not affected by the field, but causes the particle motion to form a helix along the magnetic field direction with a radius given by the Larmor radius and a pitch given by the ratio of the perpendicular to parallel velocities.

Next consider the situation in Fig. 3-1, but with the addition of a finite electric field perpendicular to **B**. In this case, **E** is in some direction in the plane of the page. The equation of motion for the charged particle is given by Eq. (3.3-1). Considering the drift to be steady-state, the time derivative is equal to zero, and Eq. (3.3-1) becomes

$$\mathbf{E} = -\mathbf{v} \times \mathbf{B}. \tag{3.3-14}$$

Taking the cross product of both sides with **B** gives

$$\mathbf{E} \times \mathbf{B} = (-\mathbf{v} \times \mathbf{B}) \times \mathbf{B} = \mathbf{v}B^2 - \mathbf{B}(\mathbf{B} \cdot \mathbf{v}). \tag{3.3-15}$$

The dot product is in the direction perpendicular to B, so the last term in Eq. (3.3-15) is equal to zero. Solving for the transverse velocity of the particle gives

$$\mathbf{v} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \equiv \mathbf{v}_E \,, \tag{3.3-16}$$

which is the "*E* cross *B*" drift velocity. In this case, the drift is in the direction perpendicular to both *E* and *B*, and arises from the cycloidal electron motion in the magnetic field being accelerated in the direction of –**E** and decelerated in the direction of **E**. This elongates the orbit on one-half cycle and shrinks the orbit on the opposite half cycle, which causes the net motion of the particle in the $\mathbf{E} \times \mathbf{B}$ direction. The units of the $\mathbf{E} \times \mathbf{B}$ velocity are

$$v_E = \frac{E\ [\mathrm{V/m}]}{B\ [\mathrm{tesla}]}\ \ (\mathrm{m/s})\,. \tag{3.3-17}$$

Finally, consider the situation of a particle gyrating in a magnetic field that is changing in magnitude along the magnetic field direction $\hat{z}$. This is commonly found in electric propulsion thrusters relatively close to permanent magnets or electromagnetic poles-pieces that produce fields used to confine the electrons. Since the divergence of *B* is zero, Eq. (3.2-3), the magnetic field in cylindrical coordinates is described by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rB_r\right) + \frac{\partial B_z}{\partial z} = 0\,. \tag{3.3-18}$$

Assuming that the axial component of the field does not vary significantly with *r* and integrating yields the radial component of the magnetic field with respect to *r*,

$$B_r \approx -\frac{r}{2}\frac{\partial B_z}{\partial z}\,. \tag{3.3-19}$$

The Lorentz force on a charged particle has a component along $\hat{z}$ given by

$$F_z \approx -q v_\phi B_r\,, \tag{3.3-20}$$

where the azimuthal particle velocity averaged over a Larmor-radius ($r = r_L$) gyration is $v_\phi = -v_\perp$. The average force on the particle is then

$$\overline{F}_z \approx -\frac{1}{2}\frac{m v_\perp^2}{B}\frac{\partial B_z}{\partial z}\,. \tag{3.3-21}$$

The magnetic moment of the gyrating particle is defined as

$$\mu = \frac{1}{2}\frac{mv_{\perp}^{2}}{B}. \qquad (3.3\text{-}22)$$

As the particle moves along the magnetic field lines into a stronger magnitude field, the parallel energy of the particle is converted into rotational energy and its Larmor radius increases. However, its magnetic moment remains invariant because the magnetic field does no work and the total kinetic energy of the particle is conserved. For a sufficiently large increase in the field, a situation can arise where the parallel velocity of the particle goes to zero and the Lorentz force reflects the particle from a "magnetic mirror." By conservation of energy, particles will be reflected from the magnetic mirror if their parallel velocity is less than

$$v_{\|} < v_{\perp}\sqrt{R_m - 1}, \qquad (3.3\text{-}23)$$

where $v_{\|}$ is the parallel velocity and R_m is the mirror ratio given by $B_{\max} / B_{\min}$. This effect is used to provide confinement of energetic electrons in ion-thruster discharge chambers.

There are a number of other particle drifts and motions possible that depend on gradients in the magnetic and electric fields, and also on time-dependent or oscillating electric or magnetic fields. These are described in detail in plasma physics texts such as Chen [1], and while they certainly might occur in the electric propulsion devices considered here, they are typically not of critical importance to the thruster performance or behavior.

3.4 Particle Energies and Velocities

In ion and Hall thrusters, the charge particles may undergo a large number of collisions with each other, and in some cases with the other species (ions, electrons, and/or neutrals) in the plasma. It is therefore impractical to analyze the motion of each particle to obtain a macroscopic picture of the plasma processes that is useful to for assessing the performance and life of these devices. Fortunately, in most cases it is not necessary to track individual particles to understand the plasma dynamics. The effect of collisions is to develop a distribution of the velocities for each species. On the average, and in the absence of other forces, each particle will then move with a speed that is solely a function of the macroscopic temperature and mass of that species. The charged particles in the thruster, therefore, can usually be described by different velocity distribution functions, and the random motions can be calculated by taking the moments of those distributions.

Most of the charged particles in electric thrusters have a Maxwellian velocity distribution, which is the most probable distribution of velocities for a group of particles in thermal equilibrium. In one dimension, the Maxwellian velocity distribution function is

$$f(v) = \left(\frac{m}{2\pi kT}\right)^{1/2} \exp\left(-\frac{mv^2}{2kT}\right), \tag{3.4-1}$$

where m is the mass of the particle, k is Boltzmann's constant, and the width of the distribution is characterized by the temperature T. The average kinetic energy of a particle in the Maxwellian distribution in one dimension is

$$E_{\text{ave}} = \frac{\int_{-\infty}^{\infty} \frac{1}{2} mv^2 \, f(v)\, dv}{\int_{-\infty}^{\infty} f(v)\, dv} . \tag{3.4-2}$$

By inserting in Eq. (3.4-1) and integrating by parts, the average energy per particle in each dimension is

$$E_{\text{ave}} = \frac{1}{2} kT . \tag{3.4-3}$$

If the distribution function is generalized into three dimensions, Eq. (3.2-8) becomes

$$f(u,v,w) = \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left[-\frac{m}{2kT}\left(u^2+v^2+w^2\right)\right], \tag{3.4-4}$$

where u, v, and w represent the velocity components in the three coordinate axes. The average energy in three dimensions is found by inserting Eq. (3.4-2) in Eq. (3.4-4) and performing the triple integration to give

$$E_{\text{ave}} = \frac{3}{2} kT . \tag{3.4-5}$$

The density of the particles is found from

$$\begin{aligned} n &= \iiint_{-\infty}^{+\infty} n f(\mathbf{v})\, d\mathbf{v} \\ &= \iiint_{-\infty}^{+\infty} n \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{m\left(u^2+v^2+w^2\right)}{2kT}\right) du dv dw . \end{aligned} \tag{3.4-6}$$

The average speed of a particle in the Maxwellian distribution is

$$\bar{v} = \int_0^\infty v \left(\frac{m}{2\pi kT} \right)^{3/2} \exp\left(-\frac{v^2}{v_{th}^2} \right) 4\pi v^2 dv \,, \qquad (3.4\text{-}7)$$

where v in Eq. (3.4-7) denotes the particle speed and v_{th} is defined as $(2kT/m)^{1/2}$. Integrating Eq. (3.4-7), the average speed per particle is

$$\bar{v} = \left(\frac{8kT}{\pi m} \right)^{1/2} . \qquad (3.4\text{-}8)$$

The flux of particles in one dimension (say in the $\hat{z}$ direction) for a Maxwellian distribution of particle velocities is given by $n < v_z >$. In this case, the average over the particle velocities is taken in the positive v_z direction because the flux is considered in only one direction. The particle flux (in one direction) is then

$$\Gamma_z = \int n v_z \, f(\mathbf{v}) d^3\mathbf{v}, \qquad (3.4\text{-}9)$$

which can be evaluated by integrating the velocities in spherical coordinates with the velocity volume element given by

$$d^3v = v^2 dv d\Omega = v^2 dv \sin\theta d\theta d\phi \,, \qquad (3.4\text{-}10)$$

where the $d\Omega$ represents the element of the solid angle. If the incident velocity has a cosine distribution ($v_z = v\cos\theta$), the one-sided flux is

$$\Gamma_z = n \left(\frac{m}{2\pi kT} \right)^{3/2} \int_0^{2\pi} d\phi \int_0^{\pi/2} \sin\theta \, d\theta \int_0^\infty v\cos\theta \; \exp\left(-\frac{v^2}{v_{th}} \right) v^2 dv \,, \quad (3.4\text{-}11)$$

which gives

$$\Gamma_z = \frac{1}{4} n\bar{v} = \frac{1}{4} n \left(\frac{8kT}{\pi m} \right)^{1/2} . \qquad (3.4\text{-}12)$$

Since the plasma electrons are very mobile and tend to make a large number of coulomb collisions with each other, they can usually be characterized by a Maxwellian temperature T_e and have average energies and speeds well described by the equations derived in this section. The random electron flux inside the plasma is also well described by Eq. (3.4-12) if the electron temperature and density are known. The electrons tend to be relatively hot (compared to the ions and atoms) in ion and Hall thrusters because they typically are injected into the plasma or heated by external mechanisms to provide sufficient energy to produce ionization. In the presence of electric and magnetic fields in the plasma and at the boundaries, the electron motion will no longer be purely random, and the flux described by Eq. (3.4-12) must be modified as described in the remainder of this chapter.

The ions in thrusters, on the other hand, are usually relatively cold in temperature (they may have high directed velocities after being accelerated, but they usually have low random velocities and temperatures). This occurs because the ions are not well confined in the plasma generators because they must be extracted to form the thrust beam, and so they leave the plasma after perhaps only a single pass. The ions are also not heated efficiently by the various mechanisms used to ionize the gas. Therefore, the plasmas in ion and Hall thrusters are usually characterized as having cold ions and Maxwellian electrons with a high electron-to-ion temperature ratio ($T_e / T_i \approx 10$). As a result, the velocity of the ions in the plasma and the fluxes to the boundaries tend to be determined by the electric fields generated inside the plasma to conserve charge, and to be different from the expressions derived here for the electron velocity and fluxes. This effect will be described in more detail in Section 3.6.

3.5 Plasma as a Fluid

The behavior of most of the plasma effects in ion and Hall thrusters can be described by simplified models in which the plasma is treated as a fluid of neutral particles and electrical charges with Maxwellian distribution functions, and the interactions and motion of only the fluid elements must be considered. Kinetic effects that consider the actual velocity distribution of each species are important in some instances, but will not be addressed here.

3.5.1 Momentum Conservation

In constructing a fluid approach to plasmas, there are three dominant forces on the charged particles in the plasma that transfer momentum that are considered here. First, charged particles react to electric and magnetic field by means of the Lorentz force, which was given by Eq. (3.3-1):

$$\mathbf{F}_L = m\frac{d\mathbf{v}}{dt} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right). \tag{3.5-1}$$

Next, there is a pressure gradient force,

$$\mathbf{F}_p = -\frac{\nabla \cdot \mathbf{p}}{n} = -\frac{\nabla\left(\mathbf{nkT}\right)}{n}, \qquad (3.5\text{-}2)$$

where the pressure is given by $P = nkT$ and should be written more rigorously as a stress tensor since it can, in general, be anisotropic. For plasmas with temperatures that are generally spatially constant, the force due to the pressure gradient is usually written simply as

$$\mathbf{F}_p = -kT\frac{\nabla \mathbf{n}}{n}. \qquad (3.5\text{-}3)$$

Finally, collisions transfer momentum between the different charged particles, and also between the charged particles and the neutral gas. The force due to collisions is

$$\mathbf{F}_c = -m\sum_{a,b} v_{ab}\left(\mathbf{v}_a - \mathbf{v}_b\right), \qquad (3.5\text{-}4)$$

where v_{ab} is the collision frequency between species *a* and *b*.

Using these three force terms, the fluid momentum equation for each species is

$$mn\frac{d\mathbf{v}}{dt} = mn\left[\frac{\partial \mathbf{v}}{\partial t} + \left(\mathbf{v}\cdot\nabla\right)\mathbf{v}\right] = qn\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right) - \nabla\cdot\mathbf{p} - mnv\left(\mathbf{v}-\mathbf{v}_o\right), \qquad (3.5\text{-}5)$$

where the convective derivative has been written explicitly and the collision term must be summed over all collisions.

Utilizing conservation of momentum, it is possible to evaluate how the electron fluid behaves in the plasma. For example, in one dimension and in the absence of magnetic fields and collisions with other species, the fluid equation of motion for electrons can be written as

$$mn_e\left[\frac{\partial v_z}{\partial t} + \left(v\cdot\nabla\right)v_z\right] = qn_eE_z - \frac{\partial p}{\partial z}, \qquad (3.5\text{-}6)$$

where v_z is the electron velocity in the z-direction and *p* represents the electron pressure term. Neglecting the convective derivative, assuming that the velocity is spatially uniform, and using Eq. (3.5-3) gives

$$m\frac{\partial v_z}{\partial t} = -eE_z - \frac{kT_e}{n_e}\frac{\partial n_e}{\partial z}. \qquad (3.5\text{-}7)$$

Assuming that the electrons have essentially no inertia (their mass is small and so they react infinitely fast to changes in potential), the left-hand side of Eq. (3.5-7) goes to zero, and the net current in the system is also zero. Considering only electrons at a temperature T_e, and using Eq. (3.2-7) for the electric field, gives

$$qE_z = e\frac{\partial\phi}{\partial z} = \frac{kT_e}{n_e}\frac{\partial n_e}{\partial z}. \tag{3.5-8}$$

Integrating this equation and solving for the electron density produces the Boltzmann relationship for electrons:

$$n_e = n_e(0)\, e^{\left(e\phi/kT_e\right)}, \tag{3.5-9}$$

where ϕ is the potential relative to the potential at the location of $n_e(0)$. Equation (3.5-9) is also sometimes known as the barometric law. This relationship simply states that the electrons will respond to electrostatic fields (potential changes) by varying their density to preserve the pressure in the system. This relationship is generally valid for motion along a magnetic field and tends to hold for motion across magnetic fields if the field is weak and the electron collisions are frequent.

3.5.2 Particle Conservation

Conservation of particles and/or charges in the plasma is described by the continuity equation:

$$\frac{\partial n}{\partial t} + \nabla \cdot n\mathbf{v} = \dot{n}_s, \tag{3.5-10}$$

where $\dot{n}_s$ represents the time-dependent source or sink term for the species being considered. Continuity equations are sometimes called mass-conservation equations because they account for the sources and sinks of particles into and out of the plasma.

Utilizing continuity equations coupled with momentum conservation and with Maxwell's equations, it is possible to calculate the response rate and wave-like behavior of plasmas. For example, the rate at which a plasma responds to changes in potential is related to the *plasma frequency* of the electrons. Assume that there is no magnetic field in the plasma or that the electron motion is along the magnetic field in the z-direction. To simplify this derivation, also assume that the ions are fixed uniformly in space on the time scales of interest here due to their large mass, and that there is no thermal motion of the particles ($T = 0$).

Since the ions are fixed in this case, only the electron equation of motion is of interest:

$$mn_e\left[\frac{\partial v_z}{\partial t}+\left(v\cdot\nabla\right)v_z\right]=-en_eE_z\,, \tag{3.5-11}$$

and the electron equation of continuity is

$$\frac{\partial n_e}{\partial t}+\nabla\cdot\left(n_e\mathbf{v}\right)=0\,. \tag{3.5-12}$$

The relationship between the electric field and the charge densities is given by Eq. (3.2-1), which for singly ionized particles can be written using Eq. (3.2-5) as

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_o}=\frac{e}{\varepsilon_o}\left(n_i-n_e\right). \tag{3.5-13}$$

The wave-like behavior of this system is analyzed by linearization using

$$\mathbf{E}=\mathbf{E}_o+\mathbf{E}_1 \tag{3.5-14}$$

$$\mathbf{v}=\mathbf{v}_o+\mathbf{v}_1 \tag{3.5-15}$$

$$n=n_o+n_1\,, \tag{3.5-16}$$

where $\mathbf{E}_o$, $\mathbf{v}_o$, and n_o are the equilibrium values of the electric field, electron velocity, and electron density, and $\mathbf{E}_1$, $\mathbf{B}_1$**,** and $\mathbf{j}_1$ are the perturbed values of these quantities. Since quasi-neutral plasma has been assumed, $\mathbf{E}_o=0$, and the assumption of a uniform plasma with no temperature means that $\Delta n_o=\mathbf{v}_o=0$. Likewise, the time derivatives of these equilibrium quantities are zero.

Linearizing Eq. (3.5-13) gives

$$\nabla\cdot\mathbf{E}_1=-\frac{e}{\varepsilon_o}n_1\,. \tag{3.5-17}$$

Using Eqs. (3.5-14), (3.5-15), and (3.5-16) in Eq. (3.5-11) results in

$$\frac{dv_1}{dt}=-\frac{e}{m}E_1\,\hat{z}\,, \tag{3.5-18}$$

where the linearized convective derivative has been neglected. Linearizing the continuity Eq. (3.5-12) gives

$$\frac{dn_1}{dt} = -n_o \nabla \cdot \mathbf{v}_1 \ \hat{z} , \tag{3.5-19}$$

where the quadratic terms, such as $n_1 v_1$, etc., have been neglected as small. In the linear regime, the oscillating quantities will behave sinusoidally:

$$\mathbf{E}_1 = E_1 \ e^{i(kz-\omega t)} \ \hat{z} \tag{3.5-20}$$

$$\mathbf{v}_1 = v_1 \ e^{i(kz-\omega t)} \ \hat{z} \tag{3.5-21}$$

$$n_1 = n_1 \ e^{i(kz-\omega t)} . \tag{3.5-22}$$

This means that the time derivates in momentum and continuity equations can be replaced by $-i\omega t$, and the gradient in Eq. (3.5-17) can be replaced by ik in the $\hat{\mathrm{z}}$ direction. Combining Eqs. (3.5-17), (3.5-18), and (3.5-19), using the time and spatial derivatives of the oscillating quantities, and solving for the frequency of the oscillation gives

$$\omega_p = \left(\frac{n_e e^2}{\varepsilon_o m} \right)^{1/2} , \tag{3.5-23}$$

where ω_p is the electron plasma frequency. A useful numerical formula for the electron plasma frequency is

$$f_p = \frac{\omega_p}{2\pi} \approx 9\sqrt{n_e} , \tag{3.5-24}$$

where the plasma density is in m^{-3}. This frequency is one of the fundamental parameters of a plasma, and the inverse of this value is approximately the minimum time required for the plasma to react to changes in its boundaries or in the applied potentials. For example, if the plasma density is 10^{18} m^{-3}, the electron plasma frequency is 9 GHz, and the electron plasma will respond to perturbations in less than a nanosecond.

In a similar manner, if the ion temperature is assumed to be negligible and the gross response of the plasma is dominated by ion motions, the ion plasma frequency can be found to be

$$\Omega_p = \left(\frac{n_e e^2}{\varepsilon_o M} \right)^{1/2} . \tag{3.5-25}$$

This equation provides the approximate time scale in which ions move in the plasma. For our previous example for a 10^{18} m^{-3} plasma density composed of xenon ions, the ion plasma frequency is about 18 MHz, and the ions will respond to first order in a fraction of a microsecond. However, the ions have inertia and respond at the ion acoustic velocity given by

$$v_a = \sqrt{\frac{\gamma_i kT_i + kT_e}{M}} , \tag{3.5-26}$$

where γ_i is the ratio of the ion specific heats and is equal to one for isothermal ions. In the normal case for ion and Hall thrusters, where $T_e >> T_i$, the ion acoustic velocity is simply

$$v_a = \sqrt{\frac{kT_e}{M}} . \tag{3.5-27}$$

It should be noted that if finite-temperature electrons and ions had been included in the derivations above, the electron-plasma and ion-plasma oscillations would have produced waves that propagate with finite wavelengths in the plasma. Electron-plasma waves and ion-plasma waves (sometimes called ion acoustic waves) occur in most electric thruster plasmas with varying amplitudes and effects on the plasma behavior. The dispersion relationships for these waves, which describe the relationship between the frequency and the wavelength of the wave, are derived in detail in plasma textbooks such as Chen [1] and will not be re-derived here.

3.5.3 Energy Conservation

The general form of the energy equation for charged species "*s*," moving with velocity $\mathbf{v}_s$ in the presence of species "*n*" is given by

$$\begin{aligned} &\frac{\partial}{\partial t}\left(n_s m_s \frac{{v_s}^2}{2} + \frac{3}{2} p_s \right) + \nabla \cdot \left(n_s m_s \frac{{v_s}^2}{2} + \frac{5}{2} p_s \right) \mathbf{v}_s + \nabla \cdot \boldsymbol{\theta}_s \\ &\qquad = q_s n_s \left(\mathbf{E} + \frac{\mathbf{R}_s}{q_s n_s} \right) \cdot \mathbf{v}_s + Q_s - \Psi_s . \end{aligned} \tag{3.5-28}$$

For simplicity, Eq. (3.5-28) neglects viscous heating of the species. The divergence terms on the left-hand side represent the total energy flux, which includes the work done by the pressure, the macroscopic energy flux, and the transport of heat by conduction $\boldsymbol{\theta}_s = \boldsymbol{\kappa}_s \Delta T_s$. The thermal conductivity of the species is denoted by $\boldsymbol{\kappa}_s$, which is given in SI units [5] by

$$\boldsymbol{\kappa}_s = 3.2 \frac{\tau_e \, ne^2 T_{eV}}{m}, \tag{3.5-29}$$

where T_{eV} in this equation is in electron volts (eV). The right-hand side of Eq. (3.5-28) accounts for the work done by other forces as well for the generation/loss of heat as a result of collisions with other particles. The term $\mathbf{R}_s$ represents the mean change in the momentum of particles "*s*" as a result of collisions with all other particles:

$$\mathbf{R}_s \equiv \sum_n \mathbf{R}_{sn} = -\sum_n n_s m_s \nu_{sn} \left(\mathbf{v}_s - \mathbf{v}_n \right). \tag{3.5-30}$$

The heat-exchange terms are Q_s, which is the heat generated/lost in the particles of species "*s*" as a result of elastic collisions with all other species, and $\boldsymbol{\Psi}_s$, the energy loss by species "*s*" as a result of inelastic collision processes such as ionization and excitation.

It is often useful to eliminate the kinetic energy from Eq. (3.5-28) to obtain a more applicable form of the energy conservation law. The left-hand side of Eq. (3.5-28) is expanded as

$$\begin{aligned} n_s m_s \mathbf{v}_s \cdot \frac{D\mathbf{v}_s}{Dt} + \frac{m_s v_s^2}{2} \frac{Dn_s}{Dt} + n_s m_s \frac{v_s^2}{2} \nabla \cdot \mathbf{v}_s + \frac{3}{2} \frac{\partial p_s}{\partial t} + \nabla \cdot \left(\frac{5}{2} p_s \mathbf{v}_s + \boldsymbol{\theta}_s \right) \\ = q_s n_s \mathbf{E} \cdot \mathbf{v}_s + \mathbf{R}_s \cdot \mathbf{v}_s + Q_s - \Psi_s. \end{aligned} \tag{3.5-31}$$

The continuity equation for the charged species is in the form

$$\frac{Dn_s}{Dt} = \frac{\partial n_s}{\partial t} + \mathbf{v}_s \cdot \nabla n_s = \dot{n} - n_s \nabla \cdot \mathbf{v}_s. \tag{3.5-32}$$

Combining these two equations with the momentum equation dotted with $\mathbf{v}_s$ gives

$$n_s m_s \mathbf{v}_s \cdot \frac{D\mathbf{v}_s}{Dt} = n_s q_s \mathbf{v}_s \cdot \mathbf{E} - \mathbf{v}_s \cdot \nabla p_s + \mathbf{v}_s \cdot \mathbf{R}_s - \dot{n} m_s v_s^2. \tag{3.5-33}$$

The energy equation can now be written as

$$\frac{3}{2}\frac{\partial p_s}{\partial t} + \nabla \cdot \left(\frac{5}{2} p_s \mathbf{v}_s + \boldsymbol{\theta}_s \right) - \mathbf{v}_s \cdot \nabla p_s = Q_s - \Psi_s - \dot{n} \frac{m_s v_s^{\ 2}}{2} . \tag{3.5-34}$$

The heat-exchange terms for each species Q_s consists of "frictional" (denoted by superscript *R*) and "thermal" (denoted by superscript *T*) contributions:

$$\begin{aligned} Q_s &= Q_s^R + Q_s^T , \\ Q_s^R &\equiv -\sum_n \mathbf{R}_{sn} \cdot \mathbf{v}_s , \\ Q_s^T &\equiv -\sum_n n_s \frac{2m_s}{m_a} \nu_{sn} \frac{3}{2}\left(\frac{kT_s}{e} - \frac{kT_n}{e} \right). \end{aligned} \tag{3.5-35}$$

In a partially ionized gas consisting of electrons, singly charged ions, and neutrals of the same species, the frictional and thermal terms for the electrons take the form

$$\begin{aligned} Q_e^R &= -\left(\mathbf{R}_{ei} + \mathbf{R}_{en} \right) \cdot \mathbf{v}_e = \left(\frac{\mathbf{R}_{ei} + \mathbf{R}_{en}}{en_e} \right) \cdot \mathbf{J}_e \\ Q_e^T &= -3n_e \frac{m}{M}\left[\nu_{ei} \frac{k}{e}\left(T_e - T_i\right) + \nu_{en} \frac{k}{e}\left(T_e - T_n\right) \right], \end{aligned} \tag{3.5-36}$$

where as usual *M* denotes the mass of the heavy species, and the temperature of the ions and neutrals is denoted by T_i and T_n, respectively. Using the steady-state electron momentum equation, in the absence of electron inertia, it is possible to write

$$Q_e^R = \left(\frac{\mathbf{R}_{ei} + \mathbf{R}_{en}}{en_e} \right) \cdot \mathbf{J}_e = \left(\mathbf{E} + \frac{\nabla p_e}{en_e} \right) \cdot \mathbf{J}_e . \tag{3.5-37}$$

Thus Eq. (3.5-34) for the electrons becomes

$$\begin{aligned} \frac{3}{2}\frac{\partial p_e}{\partial t} + \nabla \cdot \left(\frac{5}{2} p_e \mathbf{v}_e + \boldsymbol{\theta}_e \right) &= Q_e - \mathbf{J}_e \cdot \frac{\nabla p_e}{en} - \dot{n}eU_i \\ &= \mathbf{E} \cdot \mathbf{J}_e - \dot{n}eU_i , \end{aligned} \tag{3.5-38}$$

where the inelastic term is expressed as $\Psi_e = e\dot{n}U_i$ to represent the electron energy loss due to ionization, with U_i (in volts) representing the first ionization potential of the atom. In Eq. (3.5-38), the $m_e v_e^2 / 2$ correction term has been neglected because usually in ion and Hall thrusters $eU_i >> m_e v_e^2 / 2$. If multiple ionization and/or excitation losses are significant, the inelastic terms in Eq. (3.5-38) must be augmented accordingly.

In ion and Hall thrusters, it is common to assume a single temperature or distribution of temperatures for the heavy species without directly solving the energy equation(s). In some cases, however, such as in the plume of a hollow cathode for example, the ratio of T_e / T_i is important for determining the extent of Landau damping on possible electrostatic instabilities. The heavy species temperature is also important for determining the total pressure inside the cathode. Thus, separate energy equations must be solved directly. Assuming that the heavy species are slow moving and the inelastic loss terms are negligible, Eq. (3.5-34) for ions takes the form

$$\frac{3}{2}\frac{\partial p_{\mathrm{in}}}{\partial t} + \nabla \cdot \left(\frac{5}{2} p_{\mathrm{in}} \mathbf{v}_{\mathrm{in}} + \boldsymbol{\theta}_{\mathrm{in}} \right) - \mathbf{v}_{\mathrm{in}} \cdot \nabla p_{\mathrm{in}} = Q_{\mathrm{in}} , \tag{3.5-39}$$

where the subscript “in” represents ion-neutral collisions.

Finally, the total heat generated in partially ionized plasmas as a result of the (elastic) friction between the various species is given by

$$\begin{aligned} \sum_s Q_s^R &= Q_e^R + Q_i^R + Q_n^R \\ &= -\left(\mathbf{R}_{ei} + \mathbf{R}_{en}\right) \cdot \mathbf{v}_e - \left(\mathbf{R}_{ie} + \mathbf{R}_{\mathrm{in}}\right) \cdot \mathbf{v}_i - \left(\mathbf{R}_{ne} + \mathbf{R}_{ni}\right) \cdot \mathbf{v}_n . \end{aligned} \tag{3.5-40}$$

Since $\mathbf{R}_{\mathrm{sa}} = -\mathbf{R}_{\mathrm{as}}$ **,** it is possible to write this as

$$\sum_s Q_s^R = -\mathbf{R}_{ei} \cdot \left(\mathbf{v}_e - \mathbf{v}_i\right) - \mathbf{R}_{en} \cdot \left(\mathbf{v}_e - \mathbf{v}_n\right) - \mathbf{R}_{\mathrm{in}} \cdot \left(\mathbf{v}_i - \mathbf{v}_n\right) \cdot \tag{3.5-41}$$

The energy conservation equation(s) can be used with the momentum and continuity equations to provide a closed set of equations for analysis of plasma dynamics within the fluid approximations.

3.6 Diffusion in Partially Ionized Gases

Diffusion is often very important in the particle transport in ion and Hall thruster plasmas. The presence of pressure gradients and collisions between different species of charged particles and between the charged particles and the neutrals produces diffusion of the plasma from high density regions to low density regions, both along and across magnetic field lines.

To evaluate diffusion-driven particle motion in ion and Hall thruster plasmas, the equation of motion for any species can be written as

$$mn\frac{d\mathbf{v}}{dt} = qn(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \nabla \cdot \mathbf{p} - mn\nu \ (\mathbf{v} - \mathbf{v}_o), \tag{3.6-1}$$

where the terms in this equation have been previously defined and ν is the collision frequency between two species in the plasma. In order to apply and solve this equation, it is first necessary to understand the collisional processes between the different species in the plasma that determine the applicable collision frequency.

3.6.1 Collisions

Charged particles in a plasma interact with each other primarily by coulomb collisions and also can collide with neutral atoms present in the plasma. These collisions are very important when describing diffusion, mobility, and resistivity in the plasma.

When a charged particle collides with a neutral atom, it can undergo an elastic or an inelastic collision. The probability that such a collision will occur can be expressed in terms of an effective cross-sectional area. Consider a thin slice of neutral gas with an area *A* and a thickness *dx* containing essentially stationary neutral gas atoms with a density n_a. Assume that the atoms are simple spheres of cross-sectional area σ. The number of atoms in the slice is given by $n_a A dx$. The fraction of the slice area that is occupied by the spheres is

$$\frac{n_a A \sigma dx}{A} = n_a \sigma dx . \tag{3.6-2}$$

If the incident flux of particles is Γ_o, then the flux that emerges without making a collision after passing through the slice is

$$\Gamma(x) = \Gamma_o (1 - n_a \sigma dx) . \tag{3.6-3}$$

The change in the flux as the particles pass through the slice is

$$\frac{d\Gamma}{dx} = -\Gamma n_a \sigma . \tag{3.6-4}$$

The solution to Eq. (3.6-4) is

$$\Gamma = \Gamma_o \exp(-n_a \sigma\, x) = \Gamma_o \exp\left(-\frac{x}{\lambda}\right), \tag{3.6-5}$$

where λ is defined as the mean free path for collisions and describes the distance in which the particle flux would decrease to $1/e$ of its initial value. The mean free path is given by

$$\lambda = \frac{1}{n_a \sigma}, \tag{3.6-6}$$

which represents the mean distance that a relatively fast-moving particle, such as an electron or ion, will travel in a stationary density of neutral particles.

The mean time between collisions for this case is given by the mean free path divided by the charged particle velocity:

$$\tau = \frac{1}{n_a \sigma v}. \tag{3.6-7}$$

Averaging over all of the Maxwellian velocities of the charged particles, the collision frequency is then

$$\nu = \frac{1}{\tau} = n_a \sigma \bar{v}. \tag{3.6-8}$$

In the event that a relatively slowly moving particle, such as a neutral atom, is incident on a density of fast-moving electrons, the mean free path for the neutral particle to experience a collision is given by

$$\lambda = \frac{v_n}{n_e \langle \sigma v_e \rangle}, \tag{3.6-9}$$

where v_n is the neutral particle velocity and the reaction rate coefficient in the denominator is averaged over all the relevant collision cross sections. Equation (3.6-9) can be used to describe the distance that a neutral gas atom travels in a plasma before ionization occurs, which is sometimes called the *penetration distance*.

Other collisions are also very important in ion and Hall thrusters. The presence of inelastic collisions between electrons and neutrals can result in either ionization or excitation of the neutral particle. The ion production rate per unit volume is given by

$$\frac{dn_i}{dt} = n_a n_e \langle \sigma_i v_e \rangle , \tag{3.6-10}$$

where σ_i is the ionization cross section, v_e is the electron velocity, and the term in the brackets is the reaction rate coefficient, which is the ionization cross section averaged over the electron velocity distribution function.

Likewise, the production rate per unit volume of excited neutrals, n^*, is

$$\frac{dn^*}{dt} = \sum_j n_a n_e \langle \sigma_* v_e \rangle_j , \tag{3.6-11}$$

where σ_* is the excitation cross section and the reaction rate coefficient is averaged over the electron distribution function and summed over all possible excited states j. A complete listing of the ionization and excitation cross sections for xenon is given in Appendix D, and the reaction rate coefficients for ionization and excitation averaged over a Maxwellian electron distribution are given in Appendix E.

Charge exchange [2,6] in ion and Hall thrusters usually describes the resonant charge transfer between like atoms and ions in which essentially no kinetic energy is exchanged during the collision. Because this is a resonant process, it can occur at large distances, and the charge exchange (CEX) cross section is very large [2]. For example, the charge exchange cross section for xenon is about 10^{-18} m^2 [7], which is significantly larger than the ionization and excitation cross sections for this atom. Since the ions in the thruster are often energetic due to acceleration by the electric fields in the plasma or acceleration in ion thruster grid structures, charge exchange results in the production of energetic neutrals and relatively cold ions. Charge exchange collisions are often a dominant factor in the heating of cathode structures, the mobility and diffusion of ions in the thruster plasma, and the erosion of grid structures and surfaces.

While the details of classical collision physics are interesting, they are well described in several other textbooks [1,2,5] and are not critically important to understanding ion and Hall thrusters. However, the various collision frequencies and cross sections are of interest for use in modeling the thruster discharge and performance.

The frequency of collisions between electrons and neutrals is sometimes written [8] as

$$\nu_{en} = \sigma_{en}(T_e) n_a \sqrt{\frac{8kT_e}{\pi m}} \,, \tag{3.6-12}$$

where the effective electron–neutral scattering cross section $\sigma(T_e)$ for xenon can be found from a numerical fit to the electron–neutral scattering cross section averaged over a Maxwellian electron distribution [8]:

$$\sigma_{en}\left(T_e\right) = 6.6 \times 10^{-19} \left[\frac{\frac{T_{eV}}{4} - 0.1}{1 + \left(\frac{T_{eV}}{4}\right)^{1.6}} \right] \left[\mathrm{m}^2\right], \tag{3.6-13}$$

where T_{eV} is in electron volts. The electron–ion collision frequency for coulomb collisions [5] is given in SI units by

$$\nu_{ei} = 2.9 \times 10^{-12} \frac{n_e \ln \Lambda}{T_{eV}^{3/2}} \,, \tag{3.6-14}$$

where $\ln\Lambda$ is the coulomb logarithm given in a familiar form [5] by

$$\ln \Lambda = 23 - \frac{1}{2} \ln\left(\frac{10^{-6} n_e}{T_{eV}^3} \right). \tag{3.6-15}$$

The electron–electron collision frequency [5] is given by

$$\nu_{ee} = 5 \times 10^{-12} \frac{n_e \ln \Lambda}{T_{eV}^{3/2}} \,, \tag{3.6-16}$$

While the values of the electron–ion and the electron–electron collision frequencies in Eqs. (3.6-14) and (3.6-16) are clearly comparable, the electron–electron thermalization time is much shorter than the electron–ion thermalization time due to the large mass difference between the electrons and ions reducing the energy transferred in each collision. This is a major reason that electrons thermalize rapidly into a population with Maxwellian distribution, but do not thermalize rapidly with the ions.

In addition, the ion–ion collision frequency [5] is given by

$$\nu_{ii} = Z^4 \left(\frac{m}{M} \right)^{1/2} \left(\frac{T_e}{T_i} \right)^{3/2} \nu_{ee} \,, \tag{3.6-17}$$

where Z is the ion charge number.

Collisions between like particles and between separate species tend to equilibrate the energy and distribution functions of the particles. This effect was analyzed in detail by Spitzer [9] in his classic book. In thrusters, there are several equilibration time constants of interest. First, the characteristic collision times between the different charged particles is just one over the average collision frequencies given above. Second, equilibration times between the species and between different populations of the same species were calculated by Spitzer. The time for a monoenergetic electron (sometimes called a *primary electron*) to equilibrate with the Maxwellian population of the plasma electrons is called the slowing time, τ_s. Finally, the time for one Maxwellian population to equilibrate with another Maxwellian population is called the equilibration time, τ_{eq}. Expressions for these equilibration times, and a comparison of the rates of equilibration by these two effects, are found in Appendix F.

Collisions of electrons with other species in the plasma lead to resistivity and provide a mechanism for heating. This mechanism is often called ohmic heating or joule heating. In steady state and neglecting electron inertia, the electron momentum equation, taking into account electron–ion collisions and electron-neutral collisions, is

$$0 = -en\left(\mathbf{E} + \mathbf{v_e} \times \mathbf{B}\right) - \nabla \cdot \mathbf{p_e} - mn\left[\nu_{ei}\left(\mathbf{v_e} - \mathbf{v_i}\right) + \nu_{en}\left(\mathbf{v_e} - \mathbf{v_n}\right)\right]. \quad (3.6\text{-}18)$$

The electron velocity is very large with respect to the neutral velocity, and Eq. (3.6-18) can be written as

$$0 = -en\left(\mathbf{E} + \frac{\nabla \mathbf{p_e}}{en}\right) - en\mathbf{v_e} \times \mathbf{B} - mn\left(\nu_{ei} + \nu_{en}\right)\mathbf{v_e} + mn\nu_{ei}\mathbf{v_i}. \quad (3.6\text{-}19)$$

Since charged particle current density is given by $\mathbf{J} = qn\mathbf{v}$, Eq. (3.6-19) can be written as

$$\eta\mathbf{J_e} = \mathbf{E} + \frac{\nabla\mathbf{p} - \mathbf{J}_e \times \mathbf{B}}{en} - \eta_{ei}\mathbf{J_i}, \quad (3.6\text{-}20)$$

where $\mathbf{J}_e$ is the electron current density, $\mathbf{J_i}$ is the ion current density, and η_{ei} is the plasma resistivity. Equation (3.6-20) is commonly known as Ohm's law for partially ionized plasmas and is a variant of the well-known generalized Ohm's law, which usually is expressed in terms of the total current density, $\mathbf{J} = en(\mathbf{v_i} - \mathbf{v_e})$, and the ion fluid velocity, $\mathbf{v_i}$. If there are no collisions or net current in the plasma, this equation reduces to Eq. (3.5-7), which was used to derive the Boltzmann relationship for plasma electrons.

In Eq. (3.6-20), the total resistivity of a partially ionized plasma is given by

$$\eta = \frac{m(\nu_{ei} + \nu_{en})}{e^2 n} = \frac{1}{\varepsilon_o \tau_e \omega_p^2}, \tag{3.6-21}$$

where the total collision time for electrons, accounting for both electron–ion and electron–neutral collisions, is given by

$$\tau_e = \frac{1}{\nu_{ei} + \nu_{en}}. \tag{3.6-22}$$

By neglecting the electron–neutral collision terms in Eq. (3.6-19), the well-known expression for the resistivity of a fully ionized plasma [1,9] is recovered:

$$\eta_{ei} = \frac{m\nu_{ei}}{e^2 n} = \frac{1}{\varepsilon_o \tau_{ei} \omega_p^2}. \tag{3.6-23}$$

In ion and Hall thrusters, the ion current in the plasma is typically much smaller than the electron current due to the large mass ratio, so the ion current term in Ohm’s law, Eq. (3.6-20), is sometimes neglected.

3.6.2 Diffusion and Mobility Without a Magnetic Field

The simplest case of diffusion in a plasma is found by neglecting the magnetic field and writing the equation of motion for any species as

$$mn \frac{d\mathbf{v}}{dt} = qn\mathbf{E} - \nabla \cdot \mathbf{p} - mn\nu \left(\mathbf{v} - \mathbf{v}_o\right), \tag{3.6-24}$$

where m is the species mass and the collision frequency is taken to be a constant. Assume that the velocity of the particle species of interest is large compared to the slow species ($\mathbf{v} >> \mathbf{v}_o$), the plasma is isothermal ($\nabla p = kT \nabla n$), and the diffusion is steady state and occurring with a sufficiently high velocity that the convective derivative can be neglected. Equation (3.6-24) can then be solved for the particle velocity:

$$\mathbf{v} = \frac{q}{m\nu}\mathbf{E} - \frac{kT}{m\nu}\frac{\nabla n}{n}. \qquad (3.6\text{-}25)$$

The coefficients of the electric field and the density gradient terms in Eq. (3.6-25) are called the mobility,

$$\mu = \frac{|q|}{m\nu} \ [\mathrm{m^2/V\text{-}s}], \qquad (3.6\text{-}26)$$

and the diffusion coefficient,

$$D = \frac{kT}{m\nu} \ [\mathrm{m^2/s}]. \qquad (3.6\text{-}27)$$

These terms are related by what is called the *Einstein relation*:

$$\mu = \frac{|q|\, D}{kT}. \qquad (3.6\text{-}28)$$

3.6.2.1 Fick's Law and the Diffusion Equation. The flux of diffusing particles in the simple case of Eq. (3.6-25) is

$$\mathbf{\Gamma} = n\,\mathbf{v} = \mu n\,\mathbf{E} - D\,\nabla n. \qquad (3.6\text{-}29)$$

A special case of this is called *Fick's law*, in which the flux of particles for either the electric field or the mobility term being zero is given by

$$\mathbf{\Gamma} = -D\,\nabla n. \qquad (3.6\text{-}30)$$

The continuity equation, Eq. (3.5-10), without sink or source terms can be written as

$$\frac{\partial n}{\partial t} + \nabla \cdot \mathbf{\Gamma} = 0, \qquad (3.6\text{-}31)$$

where $\mathbf{\Gamma}$ represents the flux of any species of interest. If the diffusion coefficient D is constant throughout the plasma, substituting Eq. (3.6-30) into Eq. (3.6-31) gives the well-known diffusion equation for a single species:

$$\frac{\partial n}{\partial t} - D\nabla^2 n = 0. \qquad (3.6\text{-}32)$$

The solution to this equation can be obtained by separation of variables. The simplest example of this is for a slab geometry of finite width, where the plasma density can be expressed as having separable spatial and temporal dependencies:

$$n(x,t) = X(x)\overline{T}(t) . \tag{3.6-33}$$

Substituting into Eq. (3.6-32) gives

$$X \frac{d\overline{T}}{dt} = D\overline{T} \frac{d^2 X}{dx^2} . \tag{3.6-34}$$

Separating the terms gives

$$\frac{1}{\overline{T}} \frac{d\overline{T}}{dt} = D \frac{1}{X} \frac{d^2 X}{dx^2} = \alpha , \tag{3.6-35}$$

where each side is independent of the other and therefore can be set equal to a constant α. The time dependent function is then

$$\frac{d\overline{T}}{dt} = -\frac{\overline{T}}{\tau} , \tag{3.6-36}$$

where the constant α will be written as $-1/\tau$. The solution to Eq. (3.6-36) is

$$\overline{T} = \overline{T}_o e^{-t/\tau} . \tag{3.6-37}$$

Since there is no ionization source term in Eq. (3.6-32), the plasma density decays exponentially with time from the initial state.

The right-hand side of Eq. (3.6-35) has the spatial dependence of the diffusion and can be written as

$$\frac{d^2 X}{dx^2} = -\frac{X}{D\tau} , \tag{3.6-38}$$

where again the constant α will be written as $-1/\tau$. This equation has a solution of the form

$$X = A\cos\frac{X}{L} + B\sin\frac{X}{L} , \tag{3.6-39}$$

where A and B are constants and L is the diffusion length given by $(D\tau)^{1/2}$. If it is assumed that X is zero at the boundaries at $\pm d/2$, then the lowest-order solution is symmetric ($B = 0$) with the diffusion length equal to π. The solution to Eq. (3.6-38) is then

$$X = \cos\frac{\pi x}{d} . \tag{3.6-40}$$

The lowest-order complete solution to the diffusion equation for the plasma density is then the product of Eq. (3.6-37) and Eq. (3.6-40):

$$n = n_o e^{-t/\tau} \cos\frac{\pi x}{d} . \tag{3.6-41}$$

Of course, higher-order odd solutions are possible for given initial conditions, but the higher-order modes decay faster and the lowest-order mode typically dominates after a sufficient time. The plasma density decays with time from the initial value n_o, but the boundary condition (zero plasma density at the wall) maintains the plasma shape described by the cosine function in Eq. (3.6-41).

While a slab geometry was chosen for this illustrative example due to its simplicity, situations in which slab geometries are useful in modeling ion and Hall thrusters are rare. However, solutions to the diffusion equation in other coordinates more typically found in these thrusters are obtained in a similar manner. For example, in cylindrical geometries found in many hollow cathodes and in ion thruster discharge chambers, the solution to the cylindrical differential equation follows Bessel functions radially and still decays exponentially in time if source terms are not considered.

Solutions to the diffusion equation with source or sink terms on the right-hand side are more complicated to solve. This can be seen in writing the diffusion equation as

$$\frac{\partial n}{\partial t} - D\nabla^2 n = \dot{n} , \tag{3.6-42}$$

where the source term is described by an ionization rate equation given by

$$\dot{n} = n_a n \langle \sigma_i v_e \rangle \approx n_a n \sigma_i(T_e) \bar{v} , \tag{3.6-43}$$

and where $\bar{v}$ is the average particle speed found in Eq. (3.4-8) and $\sigma_i(T_e)$ is the impact ionization cross section averaged over a Maxwellian distribution of electrons at a temperature T_e. Equations for the xenon ionization reaction rate coefficients averaged over a Maxwellian distribution are found in Appendix E.

A separation of variables solution can still be obtained for this case, but the time-dependent behavior is no longer purely exponential as was found in Eq. (3.6-37). In this situation, the plasma density will decay or increase to an equilibrium value depending on the magnitude of the source and sink terms.

To find the steady-state solution to the cylindrical diffusion equation, the time derivative in Eq. (3.6-42) is set equal to zero. Writing the diffusion equation in cylindrical coordinates and assuming uniform radial electron temperatures and neutral densities, Eq. (3.6-42) becomes

$$\frac{\partial^2 n}{\partial r^2}+\frac{1}{r}\frac{\partial n}{\partial r}+\frac{\partial^2 n}{\partial z^2}+C^2 n=0, \qquad (3.6\text{-}44)$$

where the constant is given by

$$C^2=\frac{n_a\sigma_i\left(T_e\right)\bar{v}}{D}. \qquad (3.6\text{-}45)$$

This equation can be solved analytically by separation of variables of the form

$$n=n(0,0)\,f(r)g(z). \qquad (3.6\text{-}46)$$

Using Eq. (3.6-46), the diffusion equation becomes

$$\frac{1}{f}\frac{\partial^2 f}{\partial r^2}+\frac{1}{rf}\frac{\partial f}{\partial r}+C^2+\alpha^2=-\frac{1}{g}\frac{\partial^2 g}{\partial z^2}+\alpha^2=0. \qquad (3.6\text{-}47)$$

The solution to the radial component of Eq. (3.6-47) is the sum of the zero-order Bessel functions of the first and second kind, which is written in a general form as

$$f(r)=A_1 J_o\left(\alpha r\right)+A_2 Y_o\left(\alpha r\right). \qquad (3.6\text{-}48)$$

The Bessel function of the second kind, Y_o, becomes infinite as (αr) goes to zero, and because the density must always be finite, the constant A_2 must equal zero. Therefore, the solution for Eq. (3.6-47) is the product of the zero-order Bessel function of the first kind times an exponential term in the axial direction:

$$n\left(r,z\right)=n\left(0,0\right)J_o\left(\sqrt{C^2+\alpha^2}\;r\right)e^{-\alpha z}. \qquad (3.6\text{-}49)$$

Assuming that the ion density goes to zero at the wall,

$$\sqrt{C^2+\alpha^2}=\frac{\lambda_{01}}{R}, \tag{3.6-50}$$

where λ_{01} is the first zero of the zero-order Bessel function and R is the internal radius of the cylinder being considered. Setting $\alpha = 0$, this eigenvalue results in an equation that gives a direct relationship between the electron temperature, the radius of the plasma cylinder, and the diffusion rate:

$$\left(\frac{R}{\lambda_{01}}\right)^2 n_a \,\sigma_i\left(T_e\right)\sqrt{\frac{8kT_e}{\pi\, m}}-D=0\,. \tag{3.6-51}$$

The physical meaning of Eq. (3.6-51) is that particle balance in bounded plasma discharges dominated by radial diffusion determines the plasma electron temperature. This occurs because the generation rate of ions, which is determined by the electron temperature from Eq. (3.6-43), must equal the loss rate, which is determined by the diffusion rate to the walls, in order to satisfy the boundary conditions. Therefore, the solution to the steady-state cylindrical diffusion equation specifies both the radial plasma profile and the maximum electron temperature once the dependence of the diffusion coefficient is specified. This result is very useful in modeling the plasma discharges in hollow cathodes and in various types of electric thrusters.

3.6.2.2 Ambipolar Diffusion Without a Magnetic Field. In many circumstances in thrusters, the flux of ions and electrons from a given region or the plasma as a whole are equal. For example, in the case of microwave ion thrusters, the ions and electrons are created in pairs during ionization by the plasma electrons heated by the microwaves, so simple charge conservation states that the net flux of both ions and electrons out of the plasma must be the same. The plasma will then establish the required electric fields in the system to slow the more mobile electrons such that the electron escape rate is the same as the slower ion loss rate. This finite electric field affects the diffusion rate for both species.

Since the expression for the flux in Eq. (3.6-29) was derived for any species of particles, a diffusion coefficient for ions and electrons can be designated (because D contains the mass) and the fluxes equated to obtain

$$\mu_i n\mathbf{E}-D_i\ \nabla n=-\mu_e n\mathbf{E}-D_e\ \nabla n\,, \tag{3.6-52}$$

where quasi-neutrality ($n_i \approx n_e$) in the plasma has been assumed. Solving for the electric field gives

$$\mathbf{E} = \frac{D_i - D_e}{\mu_i + \mu_e} \frac{\nabla n}{n} . \tag{3.6-53}$$

Substituting **E** into Eq. (3.6-29) for the ion flux,

$$\Gamma = -\frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e} \nabla n = -D_a \nabla n , \tag{3.6-54}$$

where D_a is the ambipolar diffusion coefficient given by

$$D_a = \frac{\mu_i D_e + \mu_e D_i}{\mu_i + \mu_e} . \tag{3.6-55}$$

Equation (3.6-54) was expressed in the form of Fick's law, but with a new diffusion coefficient reflecting the impact of ambipolar flow on the particle mobilities. Substituting Eq. (3.6-54) into the continuity equation without sources or sinks gives the diffusion equation for ambipolar flow:

$$\frac{\partial n}{\partial t} - D_a \nabla^2 n = 0 . \tag{3.6-56}$$

Since the electron and ion mobilities depend on the mass

$$\mu_e = \frac{e}{m\nu} >> \mu_i = \frac{e}{M\nu} , \tag{3.6-57}$$

it is usually possible to neglect the ion mobility. In this case, Eq. (3.6-55) combined with Eq. (3.6-28) gives

$$D_a \approx D_i + \frac{\mu_i}{\mu_e} D_e = D_i \left(1 + \frac{T_e}{T_i} \right) . \tag{3.6-58}$$

Since the electron temperature in thrusters is usually significantly higher than the ion temperature ($T_e >> T_i$), ambipolar diffusion greatly enhances the ion diffusion coefficient. Likewise, the smaller ion mobility significantly decreases the ambipolar electron flux leaving the plasma.

3.6.3 Diffusion Across Magnetic Fields

Charged particle transport across magnetic fields is described by what is called *classical diffusion* theory and non-classical or *anomalous diffusion*. Classical diffusion, which will be presented below, includes both the case of particles of one species moving across the field due to collisions with another species of particles, and the case of ambipolar diffusion across the field where the fluxes are constrained by particle balance in the plasma. Anomalous diffusion can be caused by a number of different effects. In ion and Hall thrusters, the anomalous diffusion is usually described by Bohm diffusion [10].

3.6.3.1 Classical Diffusion of Particles Across B Fields. The fluid equation of motion for isothermal electrons moving in the perpendicular direction across a magnetic field is

$$mn\frac{d\mathbf{v}_\perp}{dt} = qn\left(\mathbf{E} + \mathbf{v}_\perp x\mathbf{B}\right) - kT_e\nabla n - mn\nu\ \mathbf{v}_\perp . \tag{3.6-59}$$

The same form of this equation can be written for ions with a mass M and temperature T_i. Consider steady-state diffusion and set the time and convective derivatives equal to zero. Separating Eq. (3.6-59) into x and y coordinates gives

$$mn\nu\ v_x = qnE_x + qnv_yB_o - kT_e\frac{\partial n}{\partial x} \tag{3.6-60}$$

and

$$mn\nu\ v_y = qnE_y + qnv_xB_o - kT_e\frac{\partial n}{\partial y}, \tag{3.6-61}$$

where $\mathbf{B} = B_o(z)$. The x and y velocity components are then

$$v_x = \pm\mu E_x + \frac{\omega_c}{\nu}v_y - \frac{D}{n}\frac{\partial n}{\partial x} \tag{3.6-62}$$

and

$$v_y = \pm\mu E_y + \frac{\omega_c}{\nu}v_x - \frac{D}{n}\frac{\partial n}{\partial y}. \tag{3.6-63}$$

Solving Eqs. (3.6-62) and (3.6-63), the velocities in the two directions are

$$\left[1 + \omega_c^2\tau^2\right]v_x = \pm\mu E_x - \frac{D}{n}\frac{\partial n}{\partial x} + \omega_c^2\tau^2\frac{E_y}{B_o} - \omega_c^2\tau^2\frac{kT_e}{qB_o}\frac{1}{n}\frac{\partial n}{\partial y} \tag{3.6-64}$$

and

$$\left[1+\omega_c^2\tau^2\right]v_y = \pm\mu E_y - \frac{D}{n}\frac{\partial n}{\partial y} + \omega_c^2\tau^2\frac{E_x}{B_o} - \omega_c^2\tau^2\frac{kT_e}{qB_o}\frac{1}{n}\frac{\partial n}{\partial x}, \tag{3.6-65}$$

where $\tau = 1/\nu$ is the average collision time.

The perpendicular electron mobility is defined as

$$\mu_\perp = \frac{\mu}{1+\omega_c^2\tau^2} = \frac{\mu}{1+\Omega_e^2}, \tag{3.6-66}$$

where the perpendicular mobility is written in terms of the electron Hall parameter defined as $\Omega_e = eB/m\nu$. The perpendicular diffusion coefficient is defined as

$$D_\perp = \frac{D}{1+\omega_c^2\tau^2} = \frac{D}{1+\Omega_e^2}. \tag{3.6-67}$$

The perpendicular velocity can then be written in vector form again as

$$\mathbf{v}_\perp = \pm\mu_\perp E - D_\perp\frac{\nabla n}{n} + \frac{\mathbf{v}_E + \mathbf{v}_D}{1+\left(\nu^2/\omega_c^2\right)}. \tag{3.6-68}$$

This is a form of Fick's law with two additional terms, the azimuthal $\mathbf{E}\times\mathbf{B}$ drift,

$$\mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B_o^2}, \tag{3.6-69}$$

and the diamagnetic drift,

$$\mathbf{v}_D = -\frac{kT}{qB_o^2}\frac{\nabla n\times\mathbf{B}}{n}, \tag{3.6-70}$$

both reduced by the fluid drag term ($1+\nu^2/\omega_c^2$). In the case of a thruster, the perpendicular cross-field electron flux flowing toward the wall or toward the anode is then

$$\mathbf{\Gamma}_e = n\mathbf{v}_\perp = \pm\mu_\perp n\mathbf{E} - D_\perp\nabla n, \tag{3.6-71}$$

which has the form of Fick's law but with the mobility and diffusion coefficients modified by the magnetic field.

The "classical" cross-field diffusion coefficient $D_\perp$, derived above and found in the literature [1,2], is proportional to $1/B^2$. However, in measurements in many plasma devices, including in Kaufman ion thrusters and in Hall thrusters, the perpendicular diffusion coefficient in some regions is found to be close to the Bohm diffusion coefficient:

$$D_B = \frac{1}{16}\frac{kT_e}{eB}, \tag{3.6-72}$$

which scales as $1/B$. Therefore, Bohm diffusion often progresses at orders of magnitude higher rates than classical diffusion. It has been proposed that Bohm diffusion results from collective instabilities in the plasma. Assume that the perpendicular electron flux is proportional to the $\mathbf{E} \times \mathbf{B}$ drift velocity,

$$\Gamma_e = n v_\perp \propto n\frac{E}{B}. \tag{3.6-73}$$

Also assume that the maximum electric field that occurs in the plasma due to Debye shielding is proportional to the electron temperature divided by the radius of the plasma:

$$E_{\max} = \frac{\phi_{\max}}{r} = \frac{kT_e}{qr}. \tag{3.6-74}$$

The electron flux to the wall is then

$$\Gamma_e \approx C\frac{n}{r}\frac{kT_e}{qB} \approx -C\frac{kT_e}{qB}\nabla n = -D_B \nabla n. \tag{3.6-75}$$

where C is a constant less than 1. The Bohm diffusion coefficient has an empirically determined value of C =1/16, as shown in Eq. (3.6-72), which fits most experiments with some uncertainty. As pointed out in Chen [1], this is why it is no surprise that Bohm diffusion scales as kT_e / eB.

3.6.3.2 Ambipolar Diffusion Across B Fields. Ambipolar diffusion across magnetic fields is much more complicated than the diffusion cases just covered because the mobility and diffusion coefficients are anisotropic in the presence of a magnetic field. Since both quasi-neutrality and charge balance must be satisfied, ambipolar diffusion dictates that the sum of the cross field and parallel to the field loss rates for both the ions and electrons must be the same. This means that the divergence of the ion and electron fluxes must be equal. While it is a simple matter to write equations for the divergence of these two species and equate them, the resulting equation cannot be easily solved because it depends on the behavior both in the plasma and at the boundaries conditions.

A special case in which only the ambipolar diffusion toward a wall in the presence of a transverse magnetic field is now considered. In this situation, charge balance is conserved separately along and across the magnetic field lines. The transverse electron equation of motion for isothermal electrons, including electron–neutral and electron–ion collisions, can be written as

$$mn\left(\frac{\partial \mathbf{v}_e}{\partial t}+(\mathbf{v_e}\cdot\nabla)\mathbf{v_e}\right)=-en(\mathbf{E}+\mathbf{v}_e\times\mathbf{B})-kT_e\nabla n \\ -mn\nu_{en}(\mathbf{v}_e-\mathbf{v}_o)-mn\nu_{ei}(\mathbf{v}_e-\mathbf{v}_i), \tag{3.6-76}$$

where $\mathbf{v}_o$ is the neutral particle velocity. Taking the magnetic field to be in the z-direction, and assuming the convective derivative to be negligibly small, then in steady-state this equation can be separated into the two transverse electron velocity components:

$$v_x + \mu_e E_x + \frac{e}{m\nu_e}v_y B + \frac{kT_e}{mn\nu_e}\frac{\partial n}{\partial x} - \frac{\nu_{ei}}{\nu_e}v_i = 0 \tag{3.6-77}$$

$$v_y + \mu_e E_y - \frac{e}{m\nu_e}v_x B + \frac{kT_e}{mn\nu_e}\frac{\partial n}{\partial y} - \frac{\nu_{ei}}{\nu_e}v_i = 0\ , \tag{3.6-78}$$

where $\nu_e = \nu_{en} + \nu_{ei}$ is the total collision frequency, $\mu_e = e/m\nu_e$ is the electron mobility including both ion and neutral collisional effects, and v_o is neglected as being small compared to the electron velocity v_e. Solving for v_y and eliminating the $\mathbf{E}\times\mathbf{B}$ and diagmagnetic drift terms in the x-direction, the transverse electron velocity is given by

$$v_e\left(1+\mu_e^2 B^2\right) = \mu_e\left(E + \frac{kT_e}{e}\frac{\nabla n}{n}\right) + \frac{\nu_{ei}}{\nu_e}v_i\,, \tag{3.6-79}$$

Since ambipolar flow and quasi-neutrality are assumed everywhere in the plasma, the transverse electron and ion transverse velocities must be equal, which gives

$$v_i\left(1+\mu_e^2 B^2 - \frac{\nu_{ei}}{\nu_e}\right) = \mu_e\left(E + \frac{kT_e}{e}\frac{\nabla n}{n}\right). \tag{3.6-80}$$

The transverse velocity of each species is then

$$v_i = v_e = \frac{\mu_e}{\left(1+\mu_e^2 B^2 - \dfrac{\nu_{ei}}{\nu_e}\right)}\left(E + \frac{kT_e}{e}\frac{\nabla n}{n}\right). \tag{3.6-81}$$

In this case, the electron mobility is reduced by the magnetic field (the first term on the right-hand side of this equation), and so an electric field E is generated in the plasma to actually slow down the ion transverse velocity in order to balance the pressure term and maintain ambipolarity. This is exactly the opposite of the normal ambipolar diffusion without magnetic fields or along the magnetic field lines covered in Section 3.6.2, where the electric field slowed the electrons and accelerated the ions to maintain ambipolarity. Equation (3.6-81) can be written in terms of the transverse flux as

$$\Gamma_\perp = \frac{\mu_e}{\left(1+\mu_e^2 B^2 - \nu_{ei}/\nu_e\right)}\left(enE + kT_e \nabla n\right). \tag{3.6-82}$$

3.7 Sheaths at the Boundaries of Plasmas

While the motion of the various particles in the plasma is important in understanding the behavior and performance of ion and Hall thrusters, the boundaries of the plasma represent the physical interface through which energy and particles enter and leave the plasma and the thruster. Depending on the conditions, the plasma will establish potential and density variations at the boundaries in order to satisfy particle balance or the imposed electrical conditions at the thruster walls. This region of potential and density change is called the sheath, and understanding sheath formation and behavior is also very important in understanding and modeling ion and Hall thruster plasmas.

Consider the generic plasma in Fig. 3-2, consisting of quasi-neutral ion and electron densities with temperatures given by T_i and T_e, respectively. The ion current density to the boundary "wall" for singly charged ions, to first order, is given by $n_i e v_i$, where v_i is the ion velocity. Likewise, the electron flux to the boundary wall, to first order, is given by $n_e e v_e$, where v_e is the electron velocity. The ratio of the electron flux to the ion current density going to the boundary, assuming quasi-neutrality, is

$$\frac{J_e}{J_i} = \frac{n_e e v_e}{n_i e v_i} = \frac{v_e}{v_i}. \tag{3.7-1}$$

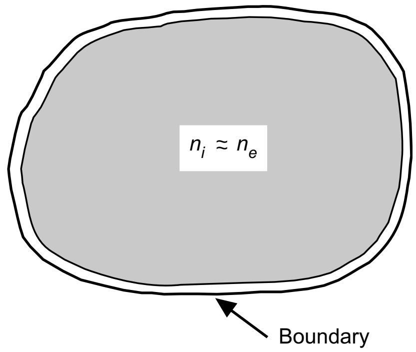

Fig. 3-2. Generic quasi-neutral plasma enclosed in a boundary.

In the absence of an electric field in the plasma volume, conservation of energy for the electrons and ions is given by

$$\frac{1}{2}mv_e^2 = \frac{kT_e}{e},$$
$$\frac{1}{2}Mv_i^2 = \frac{kT_i}{e}.$$

If it is assumed that the electrons and ions have the same temperature, the ratio of current densities to the boundary is

$$\frac{J_e}{J_i} = \frac{v_e}{v_i} = \sqrt{\frac{M}{m}}. \tag{3.7-2}$$

Table 3-1 shows the mass ratio *M/m* for several gas species. It is clear that the electron current out of the plasma to the boundary under these conditions is orders of magnitude higher than the ion current due to the much higher electron mobility. This would make it impossible to maintain the assumption of quasi-neutrality in the plasma used in Eq. (3.7-1) because the electrons would leave the volume much faster than the ions.

If different temperatures between the ions and electrons are allowed, the ratio of the current densities to the boundary is

$$\frac{J_e}{J_i} = \frac{v_e}{v_i} = \sqrt{\frac{M}{m}\frac{T_e}{T_i}}. \tag{3.7-3}$$

To balance the fluxes to the wall to satisfy charge continuity (an ionization event makes one ion and one electron), the ion temperature would have to again

Table 3-1. Ion-to-electron mass ratios for several gas species.

Gas	Mass ratio *M*/*m*	Square root of the mass ratio *M*/*m*
Protons (H^+)	1836	42.8
Argon	73440	270.9
Xenon	241066.8	490.9

be orders of magnitude higher than the electron temperatures. In ion and Hall thrusters, the opposite is true and the electron temperature is normally about an order of magnitude higher than the ion temperature, which compounds the problem of maintaining quasi-neutrality in a plasma.

In reality, if the electrons left the plasma volume faster than the ions, a charge imbalance would result due to the large net ion charge left behind. This would produce a positive potential in the plasma, which creates a retarding electric field for the electrons. The electrons would then be slowed down and retained in the plasma. Potential gradients in the plasma and at the plasma boundary are a natural consequence of the different temperatures and mobilities of the ions and electrons. Potential gradients will develop at the wall or next to electrodes inserted into the plasma to maintain quasi-neutrality between the charged species. These regions with potential gradients are called sheaths.

3.7.1 Debye Sheaths

To start an analysis of sheaths, assume that the positive and negative charges in the plasma are fixed in space, but have any arbitrary distribution. It is then possible to solve for the potential distribution everywhere using Maxwell's equations. The integral form of Eq. (3.2-1) is Gauss's law:

$$\oint_s \mathbf{E} \cdot d\mathbf{s} = \frac{1}{\varepsilon_o} \int_V \rho \, dV = \frac{Q}{\varepsilon_o} , \tag{3.7-4}$$

where Q is the total enclosed charge in the volume V and s is the surface enclosing that charge. If an arbitrary sphere of radius r is drawn around the enclosed charge, the electric field found from integrating over the sphere is

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_o r^2} \, \hat{r} . \tag{3.7-5}$$

Since the electric field is minus the gradient of the potential, the integral form of Eq. (3.2-5) can be written

$$\phi_2 - \phi_1 = -\int_{\text{p1}}^{\text{p2}} \mathbf{E} \cdot d\mathbf{l}\,, \tag{3.7-6}$$

where the integration proceeds along the path $d\mathbf{l}$ from point p1 to point p2. Substituting Eq. (3.7-5) into Eq. (3.7-6) and integrating gives

$$\phi = \frac{Q}{4\pi\varepsilon_o r}\,. \tag{3.7-7}$$

The potential decreases as $1/r$ moving away from the charge.

However, if the plasma is allowed to react to a test charge placed in the plasma, the potential has a different behavior than predicted by Eq. (3.7-7). Utilizing Eq. (3.2-7) for the electric field in Eq. (3.2-1) gives Poisson's equation:

$$\nabla^2\phi = -\frac{\rho}{\varepsilon_o} = -\frac{e}{\varepsilon_o}\left(Zn_i - n_e\right), \tag{3.7-8}$$

where the charge density in Eq. (3.2-5) has been used. Assume that the ions are singly charged and that the potential change around the test charge is small ($e\phi << kT_e$), such that the ion density is fixed and $n_i \approx n_o$. Writing Poisson's equation in spherical coordinates and using Eq. (3.5-9) to describe the Boltzmann electron density behavior gives

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = -\frac{e}{\varepsilon_o}\left[n_o - n_o \exp\left(\frac{e\phi}{kT_e}\right)\right] = \frac{en_o}{\varepsilon_o}\left[\exp\left(\frac{e\phi}{kT_e}\right) - 1\right]. \tag{3.7-9}$$

Since $e\phi << kT_e$ was assumed, the exponent can be expanded in a Taylor series:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right) = \frac{en_o}{\varepsilon_o}\left[\frac{e\phi}{kT_e} + \frac{1}{2}\left(\frac{e\phi}{kT_e}\right)^2 + \ldots\right]. \tag{3.7-10}$$

Neglecting all the higher-order terms, the solution of Eq. (3.7-10) can be written

$$\phi = \frac{e}{4\pi\varepsilon_o r}\exp\left(-r\bigg/\sqrt{\frac{\varepsilon_o kT_e}{n_o e^2}}\right). \tag{3.7-11}$$

By defining

$$\lambda_D = \sqrt{\frac{\varepsilon_o kT_e}{n_o e^2}} \tag{3.7-12}$$

as the characteristic Debye length, Eq. (3.7-12) can be written

$$\phi = \frac{e}{4\pi\varepsilon_o r} \exp\left(-\frac{r}{\lambda_D}\right). \tag{3.7-13}$$

This equation shows that the potential would normally fall off away from the test charge inserted in the plasma as $1/r$, as previously found, except that the electrons in the plasma have reacted to shield the test charge and cause the potential to decrease exponentially away from it. This behavior of the potential in the plasma is, of course, true for any structure such as a grid or probe that is placed in the plasma and that has a net charge on it.

The Debye length is the characteristic distance over which the potential changes for potentials that are small compared to kT_e. It is common to assume that the sheath around an object will have a thickness of the order of a few Debye lengths in order for the potential to fall to a negligible value away from the object. As an example, consider a plasma with a density of 10^{17} m^{-3} and an electron temperature of 1 eV. Boltzmann's constant k is 1.3807×10^{-23} J/K and the charge is 1.6022×10^{-19} coulombs, so the temperature corresponding to 1 electron volt is

$$T = 1\left(\frac{e}{k}\right) = \frac{1.6022 \times 10^{-19}}{1.3807 \times 10^{-23}} = 11604.3 \text{ K}.$$

The Debye length, using the permittivity or free space as 8.85×10^{-12} F/m is then

$$\lambda_D = \left[\frac{\left(8.85 \times 10^{-12}\right)\left(1.38 \times 10^{-23}\right)11604}{10^{17}\left(1.6 \times 10^{-19}\right)^2}\right]^{1/2}$$
$$= 2.35 \times 10^{-5} \text{ m} = 23.5 \mu\text{m}.$$

A simplifying step to note in this calculation is that kT_e / e in Eq. (3.7-12) has units of electron volts. A handy formula for the Debye length is $\lambda_D(\text{cm}) \approx 740\sqrt{T_{ev} / n_o}$, where T_{ev} is in electron volts and n_o is in cm^{-3}.

3.7.2 Pre-Sheaths

In the previous section, the sheath characteristics for the case of the potential difference between the plasma and an electrode or boundary being small compared to the electron temperature ($e\phi << kT_e$) was analyzed and resulted in Debye shielding sheaths. What happens for the case of potential differences on the order of the electron temperature? Consider a plasma in contact with a boundary wall, as illustrated in Fig. 3-3. Assume that the plasma is at a reference potential Φ at the center (which can be arbitrarily set), and that cold ions fall through an arbitrary potential of ϕ_o as they move toward the boundary. Conservation of energy states that the ions arrived at the sheath edge with an energy given by

$$\frac{1}{2}Mv_o^2 = e\phi_o . \tag{3.7-14}$$

This potential drop between the center of the plasma and the sheath edge, ϕ_o , is called the pre-sheath potential. Once past the sheath edge, the ions then gain an additional energy given by

$$\frac{1}{2}Mv^2 = \frac{1}{2}Mv_o^2 - e\phi(x) , \tag{3.7-15}$$

where v is the ion velocity in the sheath and ϕ is the potential in the sheath (becoming more negative relative to the center of the plasma). Using Eq. (3.7-14) in Eq. (3.7-15) and solving for the ion velocity in the sheath gives

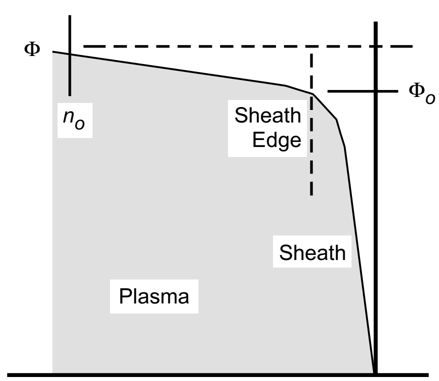

Fig. 3-3. Plasma in contact with a boundary.

$$v = \sqrt{\frac{2e}{M}}\left[\phi_o - \phi\right]^{1/2}. \tag{3.7-16}$$

However, from Eq. (3.7-14), $v_o = \sqrt{2e\phi_o/M}$, so Eq. (3.7-16) can be rearranged to give

$$\frac{v_o}{v} = \sqrt{\frac{\phi_o}{\phi_o - \phi}}, \tag{3.7-17}$$

which represents an acceleration of the ions toward the wall. The ion flux during this acceleration is conserved:

$$\begin{aligned} n_i v &= n_o v_o \\ n_i &= n_o \frac{v_o}{v}. \end{aligned} \tag{3.7-18}$$

Using Eq. (3.7-17) in Eq. (3.7-18), the ion density in the sheath is

$$n_i = n_o \sqrt{\frac{\phi_o}{\phi_o - \phi}}. \tag{3.7-19}$$

Examining the potential structure close to the sheath edge such that ϕ is small compared to the pre-sheath potential ϕ_o, Eq. (3.7-19) can be expanded in a Taylor series to give

$$n_i = n_o \left(1 - \frac{1}{2}\frac{\phi}{\phi_o} + \ldots\right), \tag{3.7-20}$$

where the higher-order terms in the series will be neglected.

The electron density through the sheath is given by the Boltzmann relationship in Eq. (3.5-9). If it is also assumed that the change in potential right at the sheath edge is small compared to the electron temperature, then the exponent in Eq. (3.5-9) can be expanded in a Taylor series to give

$$n_e = n_o \exp\left(\frac{e\phi}{kT_e}\right) = n_o\left[1 - \frac{e\phi}{kT_e} + \ldots\right]. \tag{3.7-21}$$

Using Eqs. (3.7-20) and (3.7-21) in Poisson's equation, Eq. (3.7-8), for singly charged ions in one dimension gives

$$\frac{d^2\phi}{dx^2} = -\frac{e}{\varepsilon_o}\left(n_i - n_e\right) = -\frac{en_o}{\varepsilon_o}\left[1 - \frac{1}{2}\frac{\phi}{\phi_o} - 1 + \frac{e\phi}{kT_e}\right]$$
$$= \frac{en_o\phi}{\varepsilon_o}\left[\frac{1}{2\phi_o} - \frac{e}{kT_e}\right]. \tag{3.7-22}$$

In order to avoid a positive-going inflection in the potential at the sheath edge, which would then slow or even reflect the ions going into the sheath, the right-hand side of Eq. (3.7-22) must always be positive, which implies

$$\frac{1}{2\phi_o} > \frac{e}{kT_e}. \tag{3.7-23}$$

This expression can be rewritten as

$$\phi_o > \frac{kT_e}{2e}, \tag{3.7-24}$$

which is the Bohm sheath criterion [10] that states that the ions must fall through a potential in the plasma of at least $T_e / 2$ before entering the sheath to produce a monotonically decreasing sheath potential. Since $v_o = \sqrt{2e\phi_o/M}$, Eq. (3.7-24) can be expressed in familiar form as

$$v_o \ge \sqrt{\frac{kT_e}{M}}. \tag{3.7-25}$$

This is usually called the Bohm velocity for ions entering a sheath. Equation (3.2-25) states that the ions must enter the sheath with a velocity of at least $\sqrt{kT_e / M}$ (known as the acoustic velocity for cold ions) in order to have a stable (monotonic) sheath potential behavior. The plasma produces a potential drop of at least $T_e / 2$ prior to the sheath (in the pre-sheath region) in order to produce this ion velocity. While not derived here, if the ions have a temperature T_i, it is easy to show that the Bohm velocity will still take the form of the ion acoustic velocity given by

$$v_o = \sqrt{\frac{\gamma_i kT_i + kT_e}{M}}. \tag{3.7-26}$$

It is important to realize that the plasma density decreases in the pre-sheath due to ion acceleration toward the wall. This is easily observed from the Boltzmann behavior of the plasma density. In this case, the potential at the sheath edge has fallen to a value of $-kT_e/2e$ relative to the plasma potential where the density is n_o (far from the edge of the plasma). The electron density at the sheath edge is then

$$n_e = n_o \exp\left(\frac{e\phi_o}{kT_e}\right) = n_o \exp\left[\left(\frac{e}{kT_e}\right)\left(\frac{-kT_e}{2e}\right)\right] \qquad (3.7\text{-}27)$$
$$= 0.606\ n_o .$$

Therefore, the plasma density at the sheath edge is about 60% of the plasma density in the center of the plasma.

The current density of ions entering the sheath at the edge of the plasma can be found from the density at the sheath edge in Eq. (3.7-27) and the ion velocity at the sheath edge in Eq. (3.7-25):

$$J_i = 0.6\ n_o e v_i \approx \frac{1}{2} n e \sqrt{\frac{kT_e}{M}}\,, \qquad (3.7\text{-}28)$$

where n is the plasma density at the start of the pre-sheath, which is normally considered to be the center of a collisionless plasma or one collision-mean-free path from the sheath edge for collisional plasmas. It is common to write Eq. (3.7-28) as

$$I_i = \frac{1}{2} n e \sqrt{\frac{kT_e}{M}} A\,, \qquad (3.7\text{-}29)$$

where A is the ion collection area at the sheath boundary. This current is called the Bohm current. For example, consider a xenon ion thruster with a 10^{18}m^{-3} plasma density and an electron temperature of 3 eV. The current density of ions to the boundary of the ion acceleration structure is found to be 118 A/m^2, and the Bohm current to an area of 10^{-2} m^2 is 1.18 A.

3.7.3 Child–Langmuir Sheaths

The simplest case of a sheath in a plasma is obtained when the potential across the sheath is sufficiently large that the electrons are repelled over the majority of the sheath thickness. This will occur if the potential is very large compared to the electron temperature ($\phi >> kT_e/e$). This means that the electron density goes to essentially zero relatively close to the sheath edge, and the electron space charge does not significantly affect the sheath thickness. The ion velocity through the sheath is given by Eq. (3.7-16). The ion current density is then

$$J_i = n_i e v = n_i e \sqrt{\frac{2e}{M}} \left[\phi_o - \phi\right]^{1/2} . \tag{3.7-30}$$

Solving Eq. (3.7-30) for the ion density, Poisson's equation in one dimension and with the electron density contribution neglected is

$$\frac{d^2\phi}{dx^2} = -\frac{e n_i}{\varepsilon_o} = -\frac{J_i}{\varepsilon_o}\left(\frac{M}{2e\left(\phi_o - \phi\right)}\right)^{1/2} . \tag{3.7-31}$$

The first integral can be performed by multiplying both sides of this equation by $d\phi / dx$ and integrating to obtain

$$\frac{1}{2}\left[\left(\frac{d\phi}{dx}\right)^2 - \left(\frac{d\phi}{dx}\right)^2_{x=o}\right] = \frac{2J_i}{\varepsilon_o}\left[\frac{M\left(\phi_o - \phi\right)}{2e}\right]^{1/2} . \tag{3.7-32}$$

Assuming that the electric field ($d\phi / dx$) is negligible at $x = 0$, Eq. (3.7-32) becomes

$$\frac{d\phi}{dx} = 2\left(\frac{J_i}{\varepsilon_o}\right)^{1/2}\left[\frac{M\left(\phi_o - \phi\right)}{2e}\right]^{1/4} . \tag{3.7-33}$$

Integrating this equation and writing the potential across the sheath of thickness *d* as the voltage *V* gives the familiar form of the Child–Langmuir law:

$$J_i = \frac{4\varepsilon_o}{9}\left(\frac{2e}{M}\right)^{1/2}\frac{V^{3/2}}{d^2} . \tag{3.7-34}$$

This equation was originally derived by Child [11] in 1911 and independently derived by Langmuir [12] in 1913. Equation (3.7-34) states that the current per unit area that can pass through a planar sheath is limited by space-charge effects and is proportional to the voltage to the 3/2 power divided by the sheath thickness squared. In ion thrusters, the accelerator structure can be designed to first order using the Child–Langmuir equation where *d* is the gap between the accelerator electrodes. The Child–Langmuir equation can be conveniently written as

$$\begin{aligned} J_e &= 2.33\times10^{-6}\ \frac{V^{3/2}}{d^2} \quad \text{electrons} \\ J_i &= \frac{5.45\times10^{-8}}{\sqrt{M_a}}\frac{V^{3/2}}{d^2} \quad \text{singly charged ions} \\ &= 4.75\times10^{-9}\ \frac{V^{3/2}}{d^2} \quad \text{xenon ions,} \end{aligned} \tag{3.7-35}$$

where M_a is the ion mass in atomic mass units. For example, the space-charge-limited xenon ion current density across a planar 1-mm grid gap with 1000 V applied is 15 mA/cm^2.

3.7.4 Generalized Sheath Solution

To find the characteristics of any sheath without the simplifying assumptions used in the above sections, the complete solution to Poisson's equation at a boundary must be obtained. The ion density through a planar sheath, from Eq. (3.7-19), can be written as

$$n_i = n_o\left(1-\frac{\phi}{\phi_o}\right)^{-1/2}, \tag{3.7-36}$$

and the electron density is given by Eq. (3.5-9),

$$n_e = n_o\ \exp\left(\frac{e\phi}{kT_e}\right). \tag{3.7-37}$$

Poisson's equation (3.7-8) for singly charged ions then becomes

$$\frac{d^2\phi}{dx^2} = -\frac{e}{\varepsilon_o}\left(n_i - n_e\right) = -\frac{en_o}{\varepsilon_o}\left[\left(1-\frac{\phi}{\phi_o}\right)^{-1/2} - \exp\left(\frac{e\phi}{kT_e}\right)\right]. \tag{3.7-38}$$

Defining the following dimensionless variables,

$$\chi = -\frac{e\phi}{kT_e},$$

$$\chi_o = \frac{e\phi_o}{kT_e},$$

$$\xi = \frac{x}{\lambda_D},$$

Poisson's equation becomes

$$\frac{d^2\chi}{d\xi^2} = \left(1 + \frac{\chi}{\chi_o}\right)^{-1/2} - e^{-\chi}. \qquad (3.7\text{-}39)$$

This equation can be integrated once by multiplying both sides by the first derivative of χ and integrating from $\xi_1 = 0$ to $\xi_1 = \xi$:

$$\int_0^\xi \frac{\partial\chi}{\partial\xi} \frac{\partial^2\chi}{\partial\xi^2} \, d\xi_1 = \int_0^\xi \left(1 + \frac{\chi}{\chi_o}\right)^{-1/2} \partial\chi - \int_0^\xi e^{-\chi} \, d\chi. \qquad (3.7\text{-}40)$$

where ξ_1 is a dummy variable. The solution to Eq. (3.7-40) is

$$\frac{1}{2}\left[\left(\frac{\partial\chi}{\partial\xi}\right)^2 - \left(\frac{\partial\chi}{\partial\xi}\right)^2_{\xi=0}\right] = 2\chi_o\left[\left(1 + \frac{\chi}{\chi_o}\right)^{1/2} - 1\right] + e^{-\chi} - 1. \qquad (3.7\text{-}41)$$

Since the electric field ($d\phi / dx$) is zero away from the sheath where $\xi = 0$, rearrangement of Eq. (3.7-41) yields

$$\frac{\partial\chi}{\partial\xi} = \left[4\chi_o\left(1 + \frac{\chi}{\chi_o}\right)^{1/2} + 2e^{-\chi} - 2\left(2\chi_o - 1\right)\right]^{1/2}. \qquad (3.7\text{-}42)$$

To obtain a solution for $\chi(\xi)$, Eq. (3.7-42) must be solved numerically. However, as was shown earlier for Eq. (3.7-22), the right-hand side must always be positive or the potential will have an inflection at or near the sheath edge. Expanding the right-hand side in a Taylor series and neglecting the higher-order terms, this equation will also produce the Bohm sheath criterion and specify that the ion velocity at the sheath edge must equal or exceed the ion acoustic (or Bohm) velocity. An examination of Eq. (3.7.42) shows that the Bohm sheath criterion forces the ion density to always be larger than the electron density through the pre-sheath and sheath, which results in the physically realistic monotonically decreasing potential behavior through the sheath.

Figure 3-4 shows a plot of the sheath thickness d normalized to the Debye length versus the potential drop in the sheath normalized to the electron temperature. The criterion for a Debye sheath derived in Section 3.7.1 was that the potential drop be much less than the electron temperature ($e\phi << kT_e$), which is on the far left-hand side of the graph. The criterion for a Child–Langmuir sheath derived in Section 3.7.3 is that the sheath potential be large compared to the electron temperature ($e\phi >> kT_e$), which occurs on the right-hand side of the graph. This graph illustrates the rule-of-thumb that the sheath thickness is several Debye lengths until the full Child–Langmuir conditions are established. Beyond this point, the sheath thickness varies as the potential to the 3/2 power for a given plasma density.

The reason for examining this general case is because sheaths with potential drops on the order of the electron temperature or higher are typically found at both the anode and insulating surfaces in ion and Hall thrusters. For example, it will be shown later that an insulating surface exposed to a xenon plasma will self-bias to a potential of about $6T_e$, which is called the *floating potential*. For a plasma with an electron temperature of 4 eV and a density of 10^{18}m^{-3}, the Debye length from Eq. (3.7-12) is 1.5×10^{-5}m. Since the potential is actually

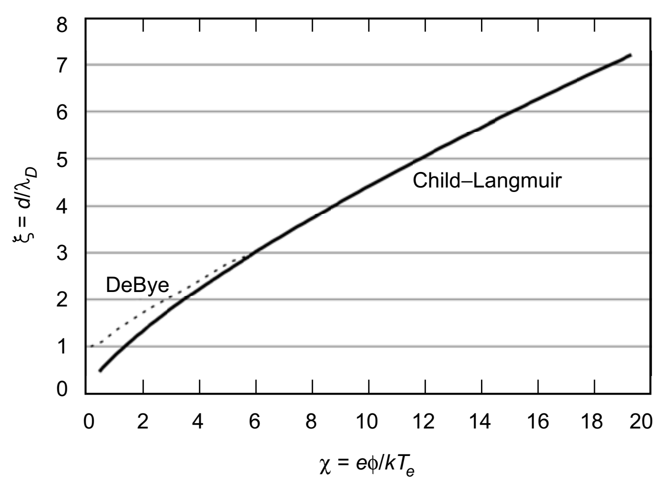

Fig. 3-4. Normalized sheath thickness as a function of the normalized sheath potential showing the transition to a Child–Langmuir sheath as the potential becomes large compared to the electron temperature.

significantly greater than the electron temperature, the sheath thickness is several times this value and the sheath transitions to a Child–Langmuir sheath.

3.7.5 Double Sheaths

So far, only plasma boundaries where particles from the plasma are flowing toward a wall have been considered. At other locations in ion and Hall thrusters, such as in some cathode and accelerator structures, a situation may exist where two plasmas are in contact but at different potentials, and ion and electron currents flow between the plasmas in opposite directions. This situation is called a double sheath, or double layer, and is illustrated in Fig. 3-5. In this case, electrons flow from the zero-potential boundary on the left, and ions flow from the boundary at a potential ϕ_s on the right. Since the particle velocities are relatively slow near the plasma boundaries before the sheath acceleration takes place, the local space-charge effects are significant and the local electric field is reduced at both boundaries. The gradient of the potential inside the double layer is therefore much higher than in the vacuum case where the potential varies linearly in between the boundaries.

Referring to Fig. 3-5, assume that the boundary on the left is at zero potential and that the particles arrive at the sheath edge on both sides of the double layer with zero initial velocity. The potential difference between the surfaces accelerates the particles in the opposite direction across the double layer. The electron conservation of energy gives

$$\begin{aligned} \frac{1}{2} m v_e^2 &= e\phi \\ v_e &= \left(\frac{2e\phi}{m} \right)^{1/2}, \end{aligned} \tag{3.7-43}$$

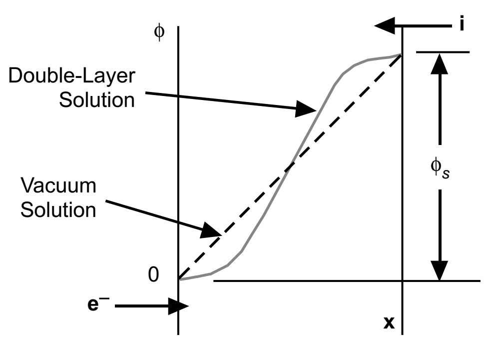

Fig. 3-5. Schematic of the double-layer potential distribution.

and the ion energy conservation gives

$$\frac{1}{2}Mv_i^2 = e\left(\phi_s - \phi\right)$$
$$v_i = \left[\frac{2e}{M}\left(\phi_s - \phi\right)\right]^{1/2} . \qquad (3.7\text{-}44)$$

The charge density in Eq. (3.2-5) can be written

$$\begin{aligned}\rho &= \rho_i + \rho_e \\ &= \frac{J_i}{v_i} - \frac{J_e}{v_e} = \frac{J_i}{\sqrt{\phi_s - \phi}}\sqrt{\frac{M}{2e}} - \frac{J_e}{\sqrt{\phi}}\sqrt{\frac{m}{2e}}.\end{aligned} \qquad (3.7\text{-}45)$$

Poisson's equation can then be written in one dimension as

$$\frac{dE}{dx} = \frac{\rho}{\varepsilon_o} = \frac{J_i}{\varepsilon_o\sqrt{\phi_s - \phi}}\sqrt{\frac{M}{2e}} - \frac{J_e}{\varepsilon_o\sqrt{\phi}}\sqrt{\frac{m}{2e}} . \qquad (3.7\text{-}46)$$

Integrating once gives

$$\frac{\varepsilon_o}{2}E^2 = 2J_i\sqrt{\frac{M}{2e}}\left[\phi_s - \left(\phi_s - \phi\right)^{1/2}\right] - 2J_e\sqrt{\frac{m}{2e}}\phi^{1/2} . \qquad (3.7\text{-}47)$$

For space-charge-limited current flow, the electric field at the right-hand boundary (the edge of the plasma) is zero and the potential is $\phi = \phi_s$. Putting that into Eq. (3.7-47) and solving for the current density gives

$$J_e = \sqrt{\frac{M}{m}}\, J_i . \qquad (3.7\text{-}48)$$

If the area of the two plasmas in contact with each other is the same, the electron current crossing the double layer is the square root of the mass ratio times the ion current crossing the layer. This situation is called the Langmuir condition (1929) and describes the space-charge-limited flow of ions and electrons between two plasmas or between a plasma and an electron emitter.

For finite initial velocities, Eq. (3.7-48) was corrected by Andrews and Allen [13] to give

$$J_e = \kappa \sqrt{\frac{M}{m}}\, J_i , \qquad (3.7\text{-}49)$$

where κ is a constant that varies from 0.8 to 0.2 for T_e / T_i changing from 2 to about 20. For typical thruster plasmas where $T_e / T_i \approx 10$, k is about 0.5.

While the presence of free-standing double layers in the plasma volume in thrusters is often debated, the sheath at a thermionic cathode surface certainly satisfies the criteria of counter-streaming ion and electron currents and can be viewed as a double layer. In this case, Eq. (3.7-49) describes the space-charge-limited current density that a plasma can accept from an electron-emitting cathode surface. This is useful in that the maximum current density that can be drawn from a cathode can be evaluated if the plasma parameters at the sheath edge in contact with the cathode are known (such that J_i can be evaluated from the Bohm current), without requiring that the actual sheath thickness be known.

Finally, there are several conditions for the formation of the classic double layer described here. In order to achieve a potential difference between the plasmas that is large compared to the local electron temperature, charge separation must occur in the layer. This, of course, violates quasi-neutrality locally. The current flow across the layer is space-charge limited, which means that the electric field is essentially zero at both boundaries. Finally, the flow through the layer discussed here is collisionless. Collisions cause resistive voltage drops where current is flowing, which can easily be confused with the potential difference across a double layer.

3.7.6 Summary of Sheath Effects

It is worthwhile to summarize here some of the important equations in this section related to sheaths because these will be very useful later in describing thruster performance. These equations were derived in the sections above, and alternative derivations can be found in [1–3].

The current density of ions entering the sheath at the edge of the plasma is given by

$$J_i = 0.6\, nev_i \approx \frac{1}{2} ne \sqrt{\frac{kT_e}{M}}\,, \tag{3.7-50}$$

where n is the plasma density at the start of the pre-sheath far from the boundary, which was considered to be the center of the plasma by Langmuir for his collisionless plasmas. The convention of approximating the coefficient 0.6 as 1/2 was made by Bohm in defining what is now called the “Bohm current.”

If there is no net current to the boundary, the ion and electron currents must be equal. The Bohm current of ions through the sheath is given by the current density in Eq. (3.7-50) times the wall area *A*:

$$I_i = \frac{1}{2} n_i \, e \sqrt{\frac{kT_e}{M}} A . \tag{3.7-51}$$

The electron current through the sheath is the random electron flux times the Boltzmann factor:

$$I_e = \frac{1}{4} \sqrt{\frac{8kT_e}{\pi m}} \, n_e \, eA \, \exp\left(-\frac{e\phi}{kT_e} \right), \tag{3.7-52}$$

where the potential is by convention a positive number in this formulation. Equating the total ion and electron currents ($I_i = I_e$), assuming quasi-neutrality in the plasma ($n_i = n_e$), and solving for the potential gives

$$\phi = \frac{kT_e}{e} \ln\left[\sqrt{\frac{2M}{\pi m}} \right]. \tag{3.7-53}$$

This is the potential at which the plasma will self-bias in order to have zero net current to the walls and thereby conserve charge and is often called the *floating potential*. Note that the floating potential is negative relative to the plasma potential.

For sheath potentials less than the electron temperature, the sheath thickness is given by the Debye length:

$$\lambda_D = \sqrt{\frac{\varepsilon_o kT_e}{n_o e^2}} . \tag{3.7-54}$$

For sheath potentials greater than the electron temperature ($e\phi > kT_e$), a pre-sheath forms to accelerate the ions into the sheath to avoid any inflection in the potential at the sheath edge. The collisionless pre-sheath has a potential difference from the center of the plasma to the sheath edge of $T_e / 2$ and a density decrease from the center of the plasma to the sheath edge of $0.61 \, n_o$. The $T_e / 2$ potential difference accelerates the ions to the Bohm velocity:

$$v_{\text{Bohm}} = v_B = \sqrt{\frac{kT_e}{M}} . \tag{3.7-55}$$

The sheath thickness at the wall depends on the plasma parameters and the potential difference between the plasma and the wall, and is found from the solution of Eq. (3.7-42).

For the case of sheath potentials that are large compared to the electron temperature ($e\phi >> kT_e$), the current density through the sheath is described by the Child–Langmuir equation:

$$J_i = \frac{4e_o}{9}\left(\frac{2e}{M}\right)^{1/2}\frac{\phi^{3/2}}{d^2}\,. \tag{3.7-56}$$

Finally, for the case of double sheaths where ion and electrons are counterstreaming across the boundary between two plasmas, the relationship between the two currents is

$$J_e \;=\; \kappa\sqrt{\frac{M}{m}}\,J_i\,. \tag{3.7-57}$$

If one boundary of the double layer is the sheath edge at a thermionic cathode, Eq. (3.7-51) can be used for the Bohm current to the opposite boundary to give the maximum emission current density as

$$J_e = \frac{\kappa}{2}n_i\,e\sqrt{\frac{kT_e}{m}} \approx \frac{1}{4}n_e\,e\sqrt{\frac{kT_e}{m}}\,. \tag{3.7-58}$$

This is the maximum electron current density that can be accepted by a plasma due to space-charge effects at the cathode double sheath. For example, the maximum space-charge-limited cathode emission current into a xenon plasma with a density of 10^{18} m^{-3} and an electron temperature of 5 eV is about 3.8 A/cm^2.

These summary equations are commonly seen in the literature on the design and analysis of ion sources, plasma processing sources, and, of course, many electric propulsion thrusters.

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Homework Problems

1. Show that Eq. (3.2-7) becomes $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ when **B** is varying with time, where **A** is the "vector potential." How are **A** and **B** related?
2. Derive Eq. (3.3-21) for the force on a particle in a magnetic mirror.
3. Show that the magnetic moment is invariant and derive Eq. (3.3-23).
4. Derive the expression for ion acoustic velocity in Eq. (3.5-26).

5. Answer the following question that might be brought up by a student working in the lab: "In a plasma discharge set up in my vacuum chamber the other day, I measured an increase in the plasma potential with an electrostatic probe. How do I know if it's a double layer or just a potential gradient within which the ionized gas is quasi-neutral?"
6. Derive Eq. (3.6-9) for the penetration distance of neutral particles in a plasma.
7. Derive the expression for Ohm's law for partially ionized plasmas, Eq. (3.6-20).
8. Derive Eq. (3.6-81) for the transverse ambipolar ion velocity across magnetic field lines.
9. Derive the Bohm sheath criteria including the presence of double ions.
10. Derive an expression equivalent to the Child–Langmuir law for the condition where the initial ion velocity entering the sheath is not neglected (ions have an initial velocity v_o at the sheath edge at $z = 0$).
11. A 2-mm by 2-mm square probe is immersed in a 3 eV xenon plasma.
    a. If the probe collects 1 mA of ion current, what is the plasma density? (Hint: the probe has two sides and is considered infinitely thin.)
    b. What is the floating potential?
    c. What is the probe current collected at the plasma potential?
12. A 2-mm-diameter cylindrical probe 5 mm long in a xenon plasma with $T_e = 3$ eV collects 1 mA of ion saturation current.
    a. What is the average plasma density?
    b. How much electron current is collected if the probe is biased to the plasma potential?
    c. Why is this electron current the same as the solution to Problem 11.c when the plasma densities are so different?
13. An electron emitter capable of emitting up to 10 A/cm^2 is in contact with an Xe^+ plasma with an electron temperature of 2 eV. Plot the emission current density versus plasma density over the range from 10^{10} to 10^{13} cm^{-3}. At what density does the emission become thermally limited (the maximum current density that the electron emitter can emit)?